ESTIMATION OF VARIATION ABOUT THE MEAN USING GEOMETRIC MEASURE

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DECLARATION

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DEDICATION

I dedicate this proposal to my grandfather Dr. Benjamin Ogweno, My Uncle Mr. Robert Okinda, Evelyne Adhiambo Odhiambo, my friend Mr. Onyango Raphael, My family and the entire Maasai Mara University Fraternity.

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ABSTRACT

A measure of dispersion is a statistical tool used to define the distribution of various datasets mainly from measures of central tendency. Some notable measures of dispersion from the mean are; average deviation, mean deviation, variance, and standard deviation. However, from previous studies, it has been established that the aforementioned measures are not absolutely perfect in estimating average variation from the mean. For instance, variance gives estimates which are of different units of measurements (squared) from the original dataset's unit of measurement. In the case of mean deviation, it gives a large average deviation than the actual deviation due to its conformation to the triangular inequality, whereas standard deviation is affected by outliers and skewed datasets. The aim of this study was to estimate variation about the mean using a technique that would overcome the weaknesses of other global measures. The study employed the geometric averaging technique to average deviation from the mean, which averages absolute products and not sums and it is nonresponsive to outliers and skewed datasets. The study formulated a geometric measure of variation for unweighted and weighted datasets, and probability mass and density functions. Using the formulations, the estimates of the average variation from the mean for the given datasets and probability distributions were computed. From the results established that the estimates obtained by the geometric measures were significantly smaller as compared to those obtained by standard deviation. In terms of efficiency, the measure was more efficient compared to standard deviation is estimating average variation about the mean for geometric, skewed and peaked datasets.

NOTATIONS

ς - weights

G_u - geometric measure for un-weighted dataset

G_w - geometric measure for weighted dataset

G_{pm} - geometric measure for probability mass function

G_{pd} - geometric measure for probability density function

G - Population geometric measure

σ - population standard deviation

g_j - sample geometric measure

s_j - sample standard deviation

$\partial_j = s_j - g_j$

d_j - the deviations

$\sum$ - Summation

ABBREVIATIONS

AD - Average Deviation

MD - Mean Deviation

SD - Standard Deviation

var - Variance

ln - Natural Logarithm

exp - Exponent

MSE - Mean Squared Error

R.E - Relative Efficiency

GM - Geometric Mean

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CHAPTER ONE

INTRODUCTION

1.1 Background

Statistical estimates such as measures of central tendency are used to give estimates which act as representative values of a given group of data. Such measures are vital tools in such fields as experimental designs where averages in determining the average response rate to an experimental treatment, in industrial production where the average estimates are used to design control charts, and in clinical trials for determining the dosage for patients (Bhardwaj, 2013; Lee et al., 2015).

The mathematical mean is the most commonly used measure of central tendency which acts as a representative measure of all observations in the dataset. The Mathematical mean is further categorized depending on the nature of the dataset as arithmetic, geometric and harmonic mean. Other measures of central tendency are median, mode and quartiles all classified as position averages. It is, therefore, important that during statistical analysis a researcher chooses an averaging technique that would give accurate results, that would objectively represent all other observations in the dataset. (Deshpande, Gogtay & Thatte, 2016; Buckland et al., 2011).

Some of the qualities of a good estimator are unbiasedness, consistency, efficiency, and sufficiency (Deshpande et al., 2016). Among these qualities, efficiency is one of the most important qualities. It is measured by the amount of variation an estimator gives during the estimation process of the true parameter value. In most fields of research, the mean is always the most preferred measure of central tendency. The efficiency of the measure is estimated by the spread of data points from the estimate, this is usually assessed through the computation of the measures of variation about the mean. During the assessment, the mean is always considered to be more precise and efficient, when most of the data points cluster around it. However, the validity of this assessment always depends on the measure of variation about mean used (Mohini & Prajakt, 2012, Roberson, Sturnman & Simon, 2007).

Currently, there are four existing measures of variation about the arithmetic mean; average deviation, mean deviation, variance, and standard deviation. Given that arithmetic mean is a representative of all the dataset, the sum of variations from the mean always add up to zero. This makes the average deviation, as a measure, to always give a result of zero. The problem of averaging the deviations from the mean has led to the development of other measures of variation about the mean such as Mean Deviation, Variance and Standard deviation all in the effort of ensuring that the total deviation from the mean do not sum up to zero (Bhardwaj, 2013; Ahn & Fessler, 2003; Atman & Bland, 2005).

Despite all the improvements that have been made on the existing measures of dispersion from the mean, further research and innovation are still underway to improve on them and even development of new measures that are more precise in estimating the average variation about the mean, so as to improve on the research precision. Following previous research findings, it has been suggested that the best way to improve on the existing measures is to make adjustments on the existing measures so as to make them more precise in estimating the average variation about the mean (Manikandan, 2011; Roberson, Sturnman & Simon, 2007). The study used this proposition to formulate a new measure that would improve on the existing measures of variation about the mean and give more precise results.

1.2 Statement of the Problem

From previous studies, it is evident that current measures of variation are not totally perfect, following the results from these studies it can be established that mean deviation violate algebraic law on absolute numbers, variance is an average of squared deviations with the squared measurement units different from the initial dataset measurement units and standard deviation is affected by outliers and skewed datasets, hence distorting the precision of the results obtained by these measures. As a result, there is a need to estimate the variation about the using other estimation techniques that might give more accurate estimate.

1.3 Objectives

1.3.1 Main Objective

The main objective of the study was to estimate the variation about the mean using the geometric measure.

1.3.2 Specific Objectives

The objectives of the study are to;

I. Formulate the geometric measure of variation about the mean.
II. Perform simulation study and compute the derived geometric measure of variation about the mean.
III. Compare the relative precision of the geometric measure and standard deviation.

1.4 Significance of the Study

The study estimated the variation about the mean for various datasets and probability distributions using the geometric measure. This was a significant contribution since the geometric measure was to overcome some of the challenges imposed by the existing measures making the estimates more precise.

1.5 Scope

The study is confined only to the use of the geometric measure of variation about the arithmetic mean and not any other measures of central tendency. In determining efficiency, the comparison was made between the geometric measures and standard deviation.

CHAPTER TWO

REVIEW OF RELATED LITERATURE

2.1 Introduction

The study is based on three critical theories, that is, Geometric Averaging and its application. Average Deviations from the Mean and Algebraic Number Theory. These theories are critical in the formulation of geometric measures and their application.

2.1.1 Geometric Average

The theory behind geometric averaging is based on the fact that, given a data vector ϑ of data points ϑ_i such that $i = 1,2,3\ldots n$, then the n^th^ root of the product of all data points ϑ_i will yield an estimate G_m which is an average representative of all the data points ϑ_i in the data vector ϑ . This can be illustrated mathematically as below (McAtister, 1879).

Given that $\vartheta = [\vartheta_1, \vartheta_2, \vartheta_3, \ldots \vartheta_n]$ such that $\vartheta_i > 0$ then the n^{th} root of the product of all data points is given as;

$$\text{Root} = \sqrt[n]{\prod \vartheta_i} = G_m \qquad (2.1)$$

Geometric averaging uses this theory to find the geometric mean for the data points (Hu, 2010; Buckland et al., 2011; Roenfeldt, 2018).

Consider a set of data points c_i such that $i = 1,2,3\ldots n$ and assume that $c_i > 0$ then the geometric mean for the data points is given by the formula

$$GM = \sqrt[n]{\prod_{i=1}^{n} c_i} \qquad (2.2)$$

Studies have found that compared to other means such as the harmonic mean and the arithmetic mean, the geometric mean always gives results which are greater than the harmonic mean but smaller than the arithmetic mean whenever all the data points are positive or greater than zero (Arithmetic > Geometric > Harmonic) (Mindlin, 2011; Hu, 2010; Roenfeldt, 2018).

Therefore, the theory shows that geometric averaging can only be used when determining the average of positive numbers and the results given by the measure are always between the arithmetic and the harmonic means. This is because if any data point is determined to be zero, then the entire results would be impossible because 0^p is undefined. Also, if the data points have p numbers that are negative given that p is odd then the results $\left(\prod_{i=1}^{n} c_i\right)^{\frac{1}{n}}$ will yield a complex result that will not be part of the data points (Hu, 2010; Buckland et al., 2011; Roenfeldt, 2018).

2.1.2 Average Deviation from the Mean

An average measure of deviation is used to measure the average distance of each data point in a distribution from the mean (Raymond, 2015). This helps in determining the distribution of the dataset. The theory behind Average Deviation from the Mean, prepositions that the magnitude of the deviation is important than the direction of the deviation. That is, a small negative distance from the mean value is better than a large positive distance because it shows higher precision than the latter. This is because datasets which compose of data points that are closer to the mean value are better than that which are composed of data points that are far away from the mean value (Grechuk, Molyboha & Zabarankin, 2011). This is because, what is important during statistical analysis is to obtain a measure of central tendency (mean) that is closer to all the data points, hence, a true representative of the entire dataset.

2.1.3 Algebraic Number Theory

This is a mathematical theory that discusses the algebraic principles that must be satisfied by algebraic numbers. The study was specifically concerned with the theory behind absolute numbers. According to the theory, an absolute number is a norm on the field P on a map $|\bullet|$ such that $|\bullet| : P \to \Re^{+}$ satisfying the following conditions

1. $|a| = 0 \ iff \ a = 0 \ \forall a \in P$
2. $|ab| = |a||b| \ \forall a, b \in P$
3. $|a+b| \leq |a| + |b| \ \forall a, b \in P$

The theory illustrates that the summation of absolute numbers results in a triangular inequality illustrated by Condition (3). However, the product of absolute numbers does not result in a difference between the estimated and the actual values, a factor that makes a product a more precise estimation technique than summation, as illustrated by Condition (2).

2.2 Critical Review

In this section, the study checked at the most current existing measures of variation about the arithmetic mean for both weighted and un-weighted datasets. In this section, the study also discusses the reasons why standard deviation is considered as the most popular measure of variation about the mean and the weaknesses of the measure based on past research.

2.2.1 Measures of Dispersion for Unweighted Dataset

Consider a dataset V such that $V = [v_1, v_2, v_3, ..., v_n]$. Such that v_i are not coefficient by any weights. The most common measures of variation about the arithmetic mean for such datasets are;

Average deviations

This is the average deviation from the mean. Using the above un-weighted data vector, the average deviation for the data is given by the formula (Raymond, 2015; Robertson, Sturnman & Simon, 2007)

$$AD = \frac{\sum_{i=1}^{n}\left(v_i - \bar{v}\right)}{n} \tag{2.3}$$

This measure of deviation from the mean for the un-weighted dataset is always known not to be appropriate since $\sum_{i=1}^{n}\left(v_i - \bar{v}\right) = 0$. As a result, this measure of variation from the mean always gives a result of 0 hence it cannot be used to assess the deviation from the mean (Deshpande, Gogtay &Thatte, 2016; Robertson, Sturnman & Simon, 2007).

Mean absolute deviation/Mean deviation

This is the average absolute deviation from the mean, it is obtained by finding the average of the absolute deviations from the mean. For the above data vector, the measure is calculated as follows (Raymond, 2015; Robertson, Sturnman & Simon, 2007)

$$MD = \frac{\sum_{i=1}^{n}\left|v_i - \bar{v}\right|}{n} \tag{2.4}$$

This measure of deviation from the mean is an improvement from the average deviation measure of deviation from the mean. However, based on the algebraic inequality, this is not an accurate measure of deviation from the mean because

$$\frac{\sum_{i=1}^{n}\left|v_i-\bar{v}\right|}{n} \geq \frac{\left|\sum_{i=1}^{n}\left(v_i-\bar{v}\right)\right|}{n}$$

Therefore, the measure always gives a bigger estimate of average variation/deviation from the mean. However, the measure makes its argument on the theory behind the estimation of average deviation, which assumes that the metric (the distance from the mean) is more important than the sign of the deviation (Deshpande, Gogtay & Thatte, 2016; Robertson, Sturnman & Simon, 2007).

Variance

This is the average measure of squared deviation from the mean. For the above un-weighted dataset, the measure is given by the formula (Raymond, 2015; Ahn & Fessler, 2003)

$$\mathrm{var} = \frac{\sum_{i=1}^{n}\left(v_i-\bar{v}\right)^2}{n} \tag{2.5}$$

where $\sum_{i=1}^{n}\left(v_i-\bar{v}\right)^2$ is the squared deviation from the mean and var is the variance.

This measure of deviation from the mean has a shortcoming of giving an average of deviations that are not of the same unit of measurement as the initial dataset (squared units). This makes the results given by the formula to be inappropriate (Manikandan, 2011; Ahn & Fessler, 2003).

Standard deviation

This is the square-root of squared deviation from the mean, it is calculated by determining the square root of variance which is the average squared deviation from the mean (Manikandan, 2011;

Bhardwaj, 2013; Mohini & Prajakt, 2012). For the above un-weighted dataset, the deviation is calculated by the formula (Raymond, 2015; Ahn & Fessler, 2003; Atman & Bland, 2005)

$$SD = \sqrt{\frac{\sum_{i=1}^{n}(v_i - \bar{v})^2}{n}} \tag{2.6}$$

where SD refers to the standard deviation.

This measure of deviation from the mean is the most common and widely used measure of deviation from the mean. Despite its wide use, the measure is faced by various problems. First, because it uses the arithmetic mean to calculate the average deviation, the measure is affected by outliers in the dataset. Second, the measure is not appropriate in measuring the deviation from the mean for the asymmetric dataset as the arithmetic averaging that it uses assumes data symmetry when averaging (Ahn & Fessler, 2003; Atman & Bland, 2005; Roberson, Sturnman & Simon, 2007).

2.2.2 Measures of Dispersion for Weighted Datasets

Consider a dataset V such that $V = [v_1, v_2, v_3, ..., v_n]$ and a set of weighted ς of the same length as V, such that $\varsigma = [\varsigma_1, \varsigma_2, \varsigma_3, ..., \varsigma_n]$. The joint distribution of the two datasets results in a weighted dataset $V\varsigma = [\varsigma_1 v_1, \varsigma_2 v_2, \varsigma_3 v_3, ... \varsigma_n v_n]$. For such dataset, the most common measures of deviation from the mean are;

Average deviations

For the above-weighted data vector, the average deviation for the data is given by the formula (Raymond, 2015; Robertson, Sturnman & Simon, 2007)

$$AD = \frac{\sum_{i=1}^{n} \varsigma_i (v_i - \bar{v})}{\sum_{i=1}^{n} \varsigma_i} \tag{2.7}$$

where $\bar{v}$ is the mean for the dataset and AD is the average deviation.

This measure of deviation from the mean for the weighted dataset is always known not to be appropriate since $\sum_{i=1}^{n} \varsigma_i (v_i - \bar{v}) = 0$. Therefore, this measure of variation from the mean for the weighted dataset always gives a result 0 hence it cannot be used to assess the deviation from the mean.

Mean absolute deviation/Mean Deviation

For the above-weighted data vector, the measure is given by (Raymond, 2015; Robertson, Sturnman & Simon, 2007)

$$MD = \frac{\sum_{i=1}^{n} \varsigma_i |v_i - \bar{v}|}{\sum_{i=1}^{n} \varsigma_i} \tag{2.8}$$

where $|v_i - \bar{v}|$ is the absolute deviation from the mean and MD refers to mean deviation

This measure of deviation from the mean for weighted datasets is an improvement from the average deviation. However, based on the algebraic triangular inequality, this is not an accurate measure of deviation from the mean because (Clark, 2012)

$$\frac{\sum_{i=1}^{n} \varsigma_i |v_i - \bar{v}|}{\sum_{i=1}^{n} \varsigma_i} > \frac{\left|\sum_{i=1}^{n} \varsigma_i (v_i - \bar{v})\right|}{\sum_{i=1}^{n} \varsigma_i}$$

Therefore, the measure always gives bigger estimates of the average variation about the mean than the actual values (Deshpande, Gogtay & Thatte, 2016; Robertson, Sturnman & Simon, 2007).

Variance

For the above-weighted dataset, the measure is given by the formula (Raymond, 2015; Ahn & Fessler, 2003)

$$\mathrm{var} = \frac{\sum_{i=1}^{n} \varsigma_i (v_i - \bar{v})^2}{\sum_{i=1}^{n} \varsigma_i} \qquad (2.9)$$

where $\sum_{i=1}^{n} \varsigma_i (v_i - \bar{v})^2$ is the sum of squared deviation from the mean and var refers to the variance

This measure of deviation from the mean for weighted datasets has a shortcoming of giving estimates which are not of the same unit as the initial dataset (squared). This makes the results given by the formula to be inappropriate in most cases as most researchers are interested in the actual deviation metrics and not squared deviation (Manikandan, 2011; Ahn & Fessler, 2003).

Standard deviation

For the above-weighted dataset, the standard deviation is calculated by (Manikandan, 2011; Bhardwaj, 2013; Mohini & Prajakt, 2012; Ahn & Fessler, 2003; Atman & Bland, 2005).

$$SD = \sqrt{\frac{\sum_{i=1}^{n} \varsigma_i (v_i - \bar{v})^2}{\sum_{i=1}^{n} \varsigma_i}} \qquad (2.10)$$

As discussed in section 2.2.2.4 the measure the most common and widely used in estimating the average variation about the mean. However, the measure is affected by outliers and skewed datasets (Leys et al., 2013).

2.2.3 Standard Deviation

Standard deviation as discussed in the previous section is a measure of deviation from the mean which gives estimates of the square-root of squared deviation from the mean also known as a variance. The measure came about as a correction of the shortcomings of variance which was the difference in unit of measurement for variance and that of the initial dataset (squared values). This problem was solved through the determination of the square root of the variance to make the results to be in the same unit of measure as the original dataset. This correction of weakness variance resulted in a new measure of the dispersion from the mean known as standard deviation (Deshpande, Gogtay & Thatte, 2016; Lee, In, & Lee; 2015; Ahn & Fessler, 2003; Atman & Bland, 2005).

Over the years, standard deviation has been popularized by its use by most researchers all over the world to the extent that it has been accepted as the ultimate measure of the dispersion from the mean. The wide use and acceptance of standard deviation have been because of the fewer number of shortcomings that are related to the measure as compared to the other existing measures of variation about the mean. Over the years, standard deviation has been considered as the superior measure of variation about the mean because the measure resulted from the improvements on the weaknesses of the existing measures of variation about the mean to the ultimate point of obtaining standard deviation (Manikandan, 2011; Bhardwaj, 2013; Mohini & Prajakt, 2012; Lee, In, and Lee; 2015).

The use of standard deviation has grown over time, from its use on un-weighted datasets to its use in weighted datasets. From its use on population datasets to its use on population samples. From its use on quantitative datasets to its use on proportions in qualitative data and also its use on distribution functions through moment generating functions. The standard deviation has also gained use in standardizations of measures and also tests of hypothesis in construction of confidence intervals through standard error and even calculation of test statistics. All these factors and improvements have enabled the measure to be recognized internationally regardless of its limitations mentioned earlier in section 2.2.2 part (d). As a result, any new measure of dispersion from the mean had to be tested against standard deviation because it is the ultimate measure of the dispersion from the mean (Manikandan, 2011; Bhardwaj, 2013; Mohini & Prajakt, 2012; Leys et al., 2013).

2.3 Summary and Gap

The review of the related literature of the existing measures of variation about the arithmetic mean show that. Mean deviation is not a precise estimator of variation about the arithmetic mean because it conforms to the triangular inequality which makes the estimates always to be bigger than the actual variation about the arithmetic mean (Deshpande, Gogtay & Thatte, 2016; Robertson, Sturnman & Simon, 2007). Variance on the other hand give estimates of squared variation about the arithmetic mean which is not of the same unit as the initial dataset (Manikandan, 2011; Ahn & Fessler, 2003). Lastly, standard deviation is effected by outliers and skewed datasets making the estimates not to be precise (Leys et al., 2013). This shows that the existing measures are not totally efficient and this gives room to estimate the variation about the arithmetic mean using other measures such as the geometric measure in this study.

CHAPTER THREE

GEOMETRIC MEASURE OF VARIATION

3.0 Introduction

The chapter describes the formulation of the geometric measure of variation about the mean for unweighted and weighted datasets, probability mass and density functions. It further discusses the techniques used in comparing the geometric measure of variation, estimates and those of standard deviation and also the techniques used in testing the efficiency of the measure in comparison to standard deviation.

3.1 Formulation of Geometric Measure

3.1.1 Formulation of Geometric Measure of Variation for un-weighted datasets

Consider a dataset $V = [v_1, v_2, v_3, v_4, ..., v_n]$, the geometric measure is constructed as follows;

Define the arithmetic mean of the vector $\bar{v}$ as

$$\bar{v} = \frac{\sum_{i=1}^{n} v_i}{n} \tag{3.1}$$

Further, define the ith deviation from the mean $\bar{v}$ as

$$d_i = v_i - \bar{v} \tag{3.2}$$

Denote the set of deviation from the mean by $d = [d_1, d_2, d_3, d_4, ..., d_n]$. Using geometric mean, define the mean of the deviation vector by

$$\bar{d}_u = \sqrt[n]{\prod_{i=1}^{n} d_i} \tag{3.3}$$

Based on the nature of mean, Equation (3.3) may not be applicable because the deviations from the mean may either be positive, negative or zero. As a result, the product of the deviations can lead to a negative number that does not have real roots. This problem can be overcome by averaging absolute deviations from the mean other than the actual deviations. Now replacing the actual deviations with absolute deviations, the mean of the deviation vector will be given by

$$\bar{d}_{Au} = \sqrt[n]{\prod_{i=1}^{n} |d_i|} \tag{3.4}$$

Equation (3.4) only applies if all the deviations $d_i \neq 0$. If any of the deviations $d_i = 0$, then the products of the absolute deviations in Equation (3.4) would yield a result of zero a factor which will make the results of the average deviations to be zero. Therefore, to overcome this challenge, it would be appropriate to only product the non-zero deviations from the mean.

Let p be the number non-zero deviations from the mean, from the deviation vector d above. The mean of the deviation vector will now be given by

$$\bar{d}_{ZAu} = \sqrt[n]{\prod_{i=1}^{p} |d_i|} \tag{3.5}$$

Let G be the geometric measure of variation about the mean. The average deviation from the mean for the above un-weighted datasets would be estimated by

$$G_u = \begin{cases} \sqrt[n]{\prod_{i=1}^{P} |d_i|} & d_1 \neq d_2 \neq d_3 \neq \ldots \neq d_n \neq 0 \\ 0 & d_1 = d_2 = d_3 = \ldots = d_n = 0 \end{cases} \tag{3.6}$$

Consider the situation in the split function in Equation (3.6) when $d_1 \neq d_2 \neq d_3 \neq \ldots \neq d_i \neq 0$, using natural logarithms, we can simplify this part of the split function as follows:

Introducing natural logarithm on both sides of the equation the product on the right-hand side would be converted into sum hence

$$\ln(G_u) = \frac{1}{n}\sum_{i=1}^{p} \ln|d_i| \tag{3.7}$$

Now, re-introducing exponential on both sides of Equation (3.7), will results in Equation (3.8) which is the geometric measure of variation about the mean under the condition $d_1 \neq d_2 \neq d_3 \neq \ldots \neq d_i \neq 0$.

$$G_u = \exp\left(\frac{1}{n}\sum_{i=1}^{p} \ln|d_i|\right) \tag{3.8}$$

Now considering both conditions when $d_1 \neq d_2 \neq d_3 \neq \ldots \neq d_n \neq 0$ and $d_1 = d_2 = d_3 = \ldots = d_n = 0$, we find out that the split function of the geometric measure is given by Equation (3.9) as;

$$G_u = \begin{cases} \exp\left(\frac{1}{n}\sum_{i=1}^{p} \ln|d_i|\right) & d_1 \neq d_2 \neq d_3 \neq \ldots \neq d_n \neq 0 \\ 0 & d_1 = d_2 = d_3 = \ldots = d_n = 0 \end{cases} \tag{3.9}$$

3.1.2 Formulation of Geometric Measure of Variation for weighted datasets

Consider a unweighted datasets $V = [v_1, v_2, v_3, \ldots, v_n]$ with the corresponding set of weights $\varsigma = [\varsigma_1, \varsigma_2, \varsigma_3, \ldots \varsigma_n]$. Define $\varsigma V = [\varsigma_1 v_1, \varsigma_2 v_2, \varsigma_3 v_3, \ldots, \varsigma_n v_n]$ as a new dataset of a weighted observations. The geometric measure of variation for the dataset is constructed as follows;

Define the arithmetic mean of the dataset $\bar{v}_w$ as

$$\bar{v}_w = \frac{\sum_{i=1}^{n} \varsigma_i v_i}{\sum_{i=1}^{n} \varsigma_i} \tag{3.10}$$

Also, define ith deviation from the mean d_{wi} as

$$d_{wi} = v_i - \bar{v}_w$$

Denote the vector deviation from the mean by $\varsigma d = \left[\varsigma_1 d_1, \varsigma_2 d_2, \varsigma_3 d_3, \ldots, \varsigma_n d_n\right]$. Using geometric mean, define the mean of the deviation vector by

$$\bar{d}_w = \sqrt[\sum_{i=1}^{n} \varsigma_i]{\prod_{i=1}^{n} d_i^{\varsigma_i}} \tag{3.11}$$

Similar to Equation (3.3), due to the equilibrium nature of mean, Equation (3.11) may not apply because the deviations from the mean may either be positive, negative or zero, as a result, the product of the deviations can lead into a negative number which does not have real roots. This can be overcome averaging absolute deviations from the mean other than the actual deviations. Now replacing the actual deviations with absolute deviations, the mean of the deviation vector will be given by

$$\bar{d}_{Aw} = \sqrt[\sum_{i=1}^{n} \varsigma_i]{\prod_{i=1}^{n} |d_i|^{\varsigma_i}} \tag{3.12}$$

Similar to Equation (3.4), Equation (3.12) only applies if all the deviations $d_i \neq 0$. If any of the deviations $d_i = 0$, then the products of the absolute deviations in Equation (3.12) would yield a result of zero a factor which will make the results of the average deviations to be zero. Therefore, to overcome this challenge, it would be appropriate to only product the non-zero deviations from the mean.

Let p be the number non-zero deviations from the mean, from the weighted deviation vector above. The mean of the deviation vector will now be given by

$$\bar{d}_{ZAw} = \sqrt[\sum_{i=1}^{n}\varsigma_i]{\prod_{i=1}^{p} |d_i|^{\varsigma_i}} \tag{3.13}$$

Let G_w be the geometric measure of variation about the mean for the weighted vector. The average deviation from the mean for weighted datasets would be estimated by

$$G_w = \begin{cases} \sqrt[\sum_{i=1}^{n}\varsigma_i]{\prod_{i=1}^{p} |d_i|^{\varsigma_i}} & d_1 \neq d_2 \neq d_3 \neq \ldots \neq d_n \neq 0 \\ 0 & d_1 = d_2 = d_3 = \ldots = d_n = 0 \end{cases} \tag{3.14}$$

Considering the split function in Equation (3.14), we can apply logarithm on the split function under the condition $d_1 \neq d_2 \neq d_3 \neq \ldots \neq d_n \neq 0$ in a similar way to Equation (3.7) and re-introduce exponential as it was done in Equation (3.8). Finally, the re-introduction of the split conditions $d_1 \neq d_2 \neq d_3 \neq \ldots \neq d_n \neq 0$ and $d_1 = d_2 = d_3 = \ldots = d_n = 0$ would result in the simplification of Equation (3.14) as illustrated in Equation (3.15).

$$G_w = \begin{cases} \exp\left(\dfrac{1}{\sum_{i=1}^{n}\varsigma_i} \sum_{i=1}^{p} \varsigma_i \ln\left(|d_i|\right) \right) & d_1 \neq d_2 \neq d_3 \neq \ldots \neq d_n \neq 0 \\ 0 & d_1 \neq d_2 \neq d_3 \neq \ldots \neq d_n \neq 0 \end{cases} \tag{3.15}$$

3.1.3 Derivation of Geometric Measure of Variation for Probability Mass Functions

Based on Equation (3.15) under condition $d_1 \neq d_2 \neq d_3 \neq \ldots \neq d_n \neq 0$ it can be shown that if all the deviation from the mean are not equal to zero, then

$$\ln(G_w) = \frac{1}{\sum_{i=1}^{n} \varsigma_i} \sum_{i=1}^{n} \varsigma_i \ln|d_i| \tag{3.16}$$

Thus, $\ln(G_w) = E(\ln|d_i|)$ and that $\ln|d_i|$ is distributed with the same weights as v_i, therefore, extending this relationship to probability mass functions. Assume that the variable v_i is discrete with probability mass function $\varsigma(v_i)$ for all $i = 1,2,3,\ldots,n$ and 0 otherwise. Assume that $\ln|d_i|$ which is equal to $\ln|v_i - \bar{v}|$ where $\bar{v}$ is the mean of the random variable V, is distributed in the same way as v_i with a probability mass function $\varsigma(v_i)$. Then the log of geometric measure $(\ln(G_{pm}))$ can be shown, based on equation (3.16), to be equivalent to

$$\ln(G_{pm}) = E(\ln|d|) \tag{3.17}$$

But $\ln|d_i|$ is distributed with probability mass function $\varsigma(v_i)$ then by definition of the expected value of probability mass function

$$\ln(G_{pm}) = E(\ln|d_i|) = \sum_{i=1}^{n} \varsigma(v_i) \ln|d_i| \quad \text{for } d_1 \neq d_2 \neq d_3 \neq \ldots \neq d_n \neq 0 \tag{3.18}$$

Hence the geometric measure for probability mass functions is be given

$$G_{pm} = \begin{cases} \exp\left(\sum_{i=1}^{n} \varsigma(v_i) \ln|d_i| \right) & d_1 \neq d_2 \neq d_3 \neq \ldots \neq d_n \neq 0 \\ 0 & d_1 = d_2 = d_3 = \ldots = d_n = 0 \end{cases} \tag{3.19}$$

3.1.4 Derivation of Geometric Measure of Variation for Probability Density Functions

Similarly, based on the conclusion made in section (3.1.3), under the relationship in Equation (3.17), we can extend the relationship on continuous random variables. Assume that the variable V is continuous on the interval $a \leq v \leq b$ with probability density function $\varsigma(v)$. Assume that $\ln|d|$ which is equal to $\ln|v-\bar{v}|$ where $\bar{v}$ is the mean of the random variable V, is distributed in the same way as V with a probability density function $\varsigma(v)$ in the same interval $a \leq v \leq b$. Using the definition of Expectation for continuous probability density functions then it can be shown that.

$$\ln(G_{pd}) = E(\ln|d|) = \int_a^b \varsigma(v)\ln|d|.dv \quad d \neq 0 \tag{3.20}$$

Hence the geometric measure for probability density functions can be given as

$$G_{pd} = \begin{cases} \exp\left(\int_a^b \varsigma(v)\ln|d|.dv\right) & d \neq 0 \\ 0 & d = 0 \end{cases} \tag{3.21}$$

All the above derivations show how the geometric measure of variation about the mean can be used to estimate the average deviation from the mean of unweighted datasets, weighted datasets, probability mass functions, and probability density functions.

3.2 Comparison of Deviation Results

The study compared the results of average deviation from the mean given by the geometric measure of deviation from the mean and standard deviation for populations of size 200 and 4000 un-weighted observations. The study simulated the observations using MINITAB version 17 software, after which the dataset was divided into 20 random samples, each of size 10 and 200 respectively, to represent small and large samples. The results of each sample did yield one geometric measure of variation from the mean and one standard deviation; hence the 20 samples did yield a total of 20 geometric measures from the mean and 20 standard deviation values for small and large samples each.

The 20 geometric measure of variation from the mean values constitute the first sample group and the 20 standard deviation values constitute the second sample values, for both small samples of size 10 and large samples of size 200. These are average deviations from the means which constitute the two samples which were considered as usual average values hence a paired sample t-test was used to determine if there is a significant difference on the two average set of values, and also, if the results for the geometric mean are significantly less than those of standard deviation.

Let $S = [s_1, s_2, s_3, s_4, ..., s_n]$ be the set of standard deviation for the 20 samples and $g = [g_1, g_2, g_3, g_4, ..., g_{20}]$ be the set of geometric measures for the 20 samples. Let $\partial_j = s_j - g_j$, which is the difference between the geometric measure and standard deviation of sample j. Let σ_∂ be the standard deviation of the difference between the geometric measure and the standard deviation of all the 20 samples. The test statistics for the paired sample t-test was given by the formula

$$t = \frac{\bar{\partial}}{\sqrt{\frac{\sigma_\partial^2}{20}}} \tag{3.22}$$

where

$$\bar{\partial} = \frac{\sum_{j=1}^{20} \partial_j}{20} \tag{3.23}$$

and

$$\sigma_{\partial}^{2} = \frac{\sum_{j=1}^{20}\left(\partial_{j} - \bar{\partial}\right)^{2}}{20} \tag{3.24}$$

A one-sided test was carried out at 95% level of confidence to determine if the estimates given by the geometric measures were significantly smaller than those given by standard deviation. The test was based on the following null and alternative hypothesis;

$$H_0 : \sigma = G$$

Against

$$H_1 : \sigma > G$$

where G is the population geometric measure while σ is the population standard deviation

The rejection of the null hypothesis, showed that the geometric measure gave smaller estimates compared to standard deviation, otherwise the test showed that the two estimates had equal estimates r standard deviation gave smaller estimates than the geometric measure.

3.3 Testing for Efficiency of the Method

The efficiency of the geometric measure of variation from the mean was tested against the standard deviation technique to determine the most efficient estimation technique of the average deviation from the mean. The study compared the efficiency of the two measures using Mean Squared Error (MSE), Bias and Relative efficiency.

3.3.1 Mean Squared Error

Consider a population of the set of data points **V** such that $V = v_1, v_2, \dots, v_N$, and let **d** be a set of deviations from the mean such that $d = d_1, d_2, \dots, d_N$ where $d_i = v_i - \bar{V}$, and $\bar{V}$ is the population mean. Let **G** be the geometric measure of variation from the mean for the population and σ be the population standard deviation.

A total of 20 samples of size 10 for small samples and size 200 for large samples were selected from a simulated population of size 200 and 4000 respectively. The population geometric measure of variation from the mean **G** and Standard deviation σ was calculated for the populations. For each sample, an estimate of average variation about the mean was calculated using the geometric measure of variation from the mean and also using standard deviation. This yielded a total of 20 estimates for the geometric measure and similar 20 estimates for the standard deviation for both small and large samples. The average of the 20 geometric measure estimates was given by

$$\bar{g} = \frac{\sum_{j=1}^{20} g_j}{20} \tag{3.25}$$

where g_i is the geometric measure of variation from the mean for the jth sample which was given by the formula

$$g_j = \begin{cases} \exp(\frac{1}{n}\sum_{i=1}^{P} \ln(|d_i|)) & d_1 \neq d_2 \neq d_3 \neq \dots \neq d_n \neq 0 \\ 0 & d_1 = d_2 = d_3 = \dots = d_n = 0 \end{cases} \tag{3.26}$$

Similarly, the mean of the standard deviation estimators was given by

$$\bar{S} = \frac{\sum_{j=1}^{20} S_j}{20} \tag{3.27}$$

where,

$$S_j = \frac{\sum_{i=1}^{n} \left(v_i - \bar{v}\right)^2}{n-1} \tag{3.28}$$

The mean squared error for the geometric measure of variation from the mean was given by

$$MSE = \frac{\sum_{j=1}^{20} \left(g_j - G\right)^2}{20} \tag{3.29}$$

where

$$G = \begin{cases} \exp\left(\frac{1}{N}\sum_{i=1}^{P} \ln(|d_i|)\right) & d_1 \neq d_2 \neq d_3 \neq \ldots \neq d_n \neq 0 \\ 0 & d_1 = d_2 = d_3 = \ldots = d_n = 0 \end{cases} \tag{3.30}$$

Similarly, for standard deviation estimates the mean squared error was given by;

$$MSE = \frac{\sum_{j=1}^{20} \left(S_j - \sigma\right)^2}{20} \tag{3.31}$$

Where,

$$\sigma = \sqrt{\frac{\sum_{i=1}^{N} \left(v_i - \bar{v}\right)^2}{N}} \tag{3.32}$$

The MSE results for Equation (3.29) was compared to that of Equation (3.31). The estimation technique that gave smaller MSE was considered to be a more efficient estimator than the other.

3.3.2 Bias

Bias refers to the difference between the estimated value and the parameter value in point estimation. During the estimation of the average from the mean using either the geometric measure of deviation from the mean or standard deviation, the bias was given by the difference between the population parameters G and σ, against the average of the sample estimates $\bar{g}$ and $\bar{S}$, respectively. For the geometric measure of variation about the mean, the bias was given by;

$$BIAS = \bar{g} - G \tag{3.33}$$

Similarly, for standard deviation, the bias was given by;

$$BIAS = \bar{S} - \sigma \tag{3.34}$$

The estimation method which had smaller bias based on Equation (3.33) and (3.34) was deemed more efficient than the other.

3.3.4 Relative Efficiency

The efficiency of the geometric measure of variation was compared to that of standard deviation using the relative efficiency. Relative efficiency was given as the ratio of the MSE of the two measures. Based on the MSE estimates given by Equation (3.29) and Equation (3.31), the relative efficiency was given by;

$$\text{Relative Efficiency} = \frac{MSE(S)}{MSE(g)} \tag{3.35}$$

When the result of Equation (3.35) was determined to be less than 1 then the standard deviation was deemed more efficient than geometric measure. However, when the result was determined to be more than 1 standard deviation was considered to be less efficient compared to the geometric measure of variation. Otherwise, when the result was 1 then the two measures were deemed to be equally efficient in estimating the average variation about the mean.

CHAPTER FOUR

APPLICATION AND EFFICIENCY TEST

4.0 Introduction

This chapter illustrates the application of the geometric measure in estimating the average variation about the mean for various simulated datasets. The chapter also gives results on the comparison of estimates of the geometric measure and standard deviation for various simulated datasets. Lastly, the study concludes by discussing the efficiency results of the geometric measures in comparison to standard deviation for various simulated datasets.

4.1 Application of Geometric Measure

4.1.1. Application on Un-Weighted Datasets

Several simulations were conducted on both discrete and continuous data distributions of small samples of size 10 from Bernoulli, Binomial, Geometric, Normal, chi-square and F-distributed populations. The results were as shown below;

4.1.1.1 Bernoulli Sample

Consider a sample of size 10 from a Bernoulli distributed population with a probability of success of 0.7. The average variation about the mean for the sample was estimated using geometric measure as shown in Table 4.1;

Table 4.1: Application on small Bernoulli sample

Observation Number	Bernoulli Sample	$d_i = x_i - 0.8$	$\|d_i\|$	$\ln(\|d_i\|)$
1	1	0.2	0.2	-1.609437912
2	1	0.2	0.2	-1.609437912
3	1	0.2	0.2	-1.609437912

4	1	0.2	0.2	-1.609437912
5	0	-0.8	0.8	-0.223143551
6	1	0.2	0.2	-1.609437912
7	1	0.2	0.2	-1.609437912
8	1	0.2	0.2	-1.609437912
9	0	-0.8	0.8	-0.223143551
10	1	0.2	0.2	-1.609437912
Mean	0.8			-1.33217904

Geometric measure $= \exp(-1.33217904) = 0.263901582 \approx 0.2639$

4.1.1.2 Binomial Sample

Consider a sample of size 10 from a Binomially distributed population with 30 trials and a probability of success of 0.64. The average variation about the mean for the sample was estimated using geometric measure as shown in Table 4.2;

Table 4.2: Application on small Binomial sample

Observation Number	Binomial Sample	$d_i = x_i - 18.2$	$\lvert d_i \rvert$	$\ln(\lvert d_i \rvert)$
1	18	-0.2	0.2	-1.609437912
2	17	-1.2	1.2	0.182321557
3	20	1.8	1.8	0.587786665
4	18	-0.2	0.2	-1.609437912
5	15	-3.2	3.2	1.16315081
6	16	-2.2	2.2	0.78845736
7	23	4.8	4.8	1.568615918
8	17	--1.2	1.2	0.182321557
9	20	1.8	1.8	0.587786665
10	18	-0.2	0.2	-1.609437912
Mean	18.2			0.023212679

Geometric measure $= \exp(0.023212679) = 1.02348419 \approx 1.0235$

4.2.1.3. Geometric Sample

Consider a sample of size 10 from a Geometrically distributed population with a probability of success of 0.5. The average variation about the mean for the sample was estimated using geometric measure as illustrated in Table 4.3;

Table 4.3: Application on small geometric sample

Observation Number	Geometric Sample	$d_i = x_i - 1.9$	$\lvert d_i \rvert$	$\ln(\lvert d_i \rvert)$
1	1	-0.9	0.9	-0.105360516
2	1	-0.9	0.9	-0.105360516
3	2	0.1	0.1	-2.302585093
4	3	1.1	1.1	0.09531018
5	1	-0.9	0.9	-0.105360516
6	1	-0.9	0.9	-0.105360516
7	1	-0.9	0.9	-0.105360516
8	3	1.1	1.1	0.09531018
9	1	-0.9	0.9	-0.105360516
10	5	3.1	3.1	1.131402111
Mean	1.9			-0.161272572

Geometric measure $= \exp(-0.161272572) = 0.851060065 \approx 0.8511$

4.1.1.4 Chi-square Sample

Sample of size 10 from a Chi-square distributed population with 1 degree of freedom was simulated as shown in Table 4.4. The average variation about the mean for the sample was estimated using geometric measure as illustrated in Table 4.4;

Table 4.4: Application on Small Chi-square sample

Observation Number	Chi-square Sample	$d_i = x_i - 1.580066995$	$\lvert d_i \rvert$	$\ln(\lvert d_i \rvert)$
1	3.217024949	1.636957954	1.636957954	0.492839614
2	2.995690441	1.415623446	1.415623446	0.347570032
3	3.662891652	2.082824657	2.082824657	0.733724981
4	1.481728301	-0.098338694	0.098338694	-2.319337698

5	2.378023987	0.797956993	0.797956993	-0.225700577
6	0.189166369	-1.390900625	1.390900625	0.329951469
7	0.093799897	-1.486267098	1.486267098	0.396267673
8	1.601235263	0.021168268	0.021168268	-3.855251998
9	0.180945899	-1.399121096	1.399121096	0.335844251
10	0.000163189	-1.579903805	1.579903805	0.457363962
Mean	1.580066995			-0.330672829

Geometric measure $= \exp(-0.330672829) = 0.718440183 \approx 0.7184$

4.1.1.5 Normal Distribution Sample

A sample of size 10 from a Normal distribution population with a mean of 10 and a standard deviation of 2 was simulated as illustrated in Table 4.5. The average variation about the mean for the sample was estimated using geometric measure as illustrated in Table 4.5;

Table 4.5: Application on small Normal Distribution Sample

Observation Number	Normal Sample	$d_i = x_i - 9.560952744$	$\lvert d_i \rvert$	$\ln(\lvert d_i \rvert)$
1	9.4732888	-0.087663944	0.087663944	-2.434244594
2	10.0179588	0.457006053	0.457006053	-0.783058643
3	9.833743354	0.272790611	0.272790611	-1.299050771
4	9.843723505	0.282770761	0.282770761	-1.263118741
5	7.86179902	-1.699153723	1.699153723	0.530130317
6	10.54505528	0.98410254	0.98410254	-0.01602518
7	8.855622899	-0.705329844	0.705329844	-0.349089721
8	9.473922934	-0.087029809	0.087029809	-2.441504582
9	13.65741087	4.096458131	4.096458131	1.41012273
10	6.047001969	-3.513950775	3.513950775	1.256740981
Mean	9.560952744			-0.53890982

Geometric measure $= \exp(-0.53890982) = 0.583383899 \approx 0.5834$

4.1.1.6 F-distributed Sample

A sample of size 10 from an F-distribution population with 2 numerator and 5 denominator degrees of freedom was simulated as illustrated in Table 4.6. The average variation about the mean for the sample was estimated using geometric measure as illustrated in Table 4.6;

Table 4.6: Application on small F-distributed sample

Observation Number	F-distribution Sample	$d_i = x_i - 2.2730346666$	$\lvert d_i \rvert$	$\ln(\lvert d_i \rvert)$
1	7.650678068	5.377643402	5.377643402	1.682250249
2	2.329789864	0.056755198	0.056755198	-2.869008029
3	5.536239586	3.26320492	3.26320492	1.182709817
4	1.738601369	-0.534433297	0.534433297	-0.626548351
5	0.771963586	-1.50107108	1.50107108	0.406178907
6	0.555808899	-1.717225767	1.717225767	0.540710062
7	1.502359971	-0.770674695	0.770674695	-0.260488921
8	0.31742318	-1.955611486	1.955611486	0.670702925
9	0.975575768	-1.297458898	1.297458898	0.260407658
10	1.351906368	-0.921128298	0.921128298	-0.082155949
Mean	2.273034666			0.090475837

Geometric measure $= \exp(0.0904758837) = 1.0946950556 \approx 1.0947$

Based on the illustration shown by the calculations, the geometric measure geometric measure was able to be used to estimate variation about the mean for selected number of un-weighted discrete and continuous datasets.

4.1.2 Application on Weighted Datasets

The study intended to determine if the geometric measure could be used in the estimation of average deviations from the mean for weighted datasets with specific concentration on frequency distributions, where the frequencies were used as the weights of the respective data points. Simulated data for three discrete distributions (Bernoulli, Binomial and Geometric distributions) and three continuous distributions (Normal, Chi-square and Fishers/F-distributions) were used. 100 observations were simulated for each distribution after which the data were summarized into frequency distributions before geometric measure for the average deviation about the mean was estimated for each frequency distribution. The result of the simulation and the estimation is as illustrated below;

4.1.2.1 Bernoulli sample

Consider a sample of size 100 from Bernoulli distributed population with a probability of success 0.7. The frequency distribution and the estimation of average variation about the mean using geometric measure were as illustrated in Table 4.7;

Table 4.7: Application on weighted Bernoulli sample

Number (v_i)	Frequency (ς_i)	$\varsigma_i v_i$	$d_i = v_i - \bar{v}$	$\lvert v_i - \bar{v}\rvert$	$\ln\lvert d_i\rvert = \ln\lvert v_i - \bar{v}\rvert$	$\varsigma_i \ln\lvert d_i\rvert = \varsigma_i \lvert v_i - \bar{v}\rvert$
0	34	0	-0.66	0.66	-0.415515444	-14.12752509
1	66	66	0.34	0.34	-1.078809661	-71.20143765
Total	100	66				-85.32896275

$$Mean(\bar{v}) = \frac{\sum_{i=1}^{n} \varsigma_i v_i}{\sum_{i=1}^{n} \varsigma_i} = \frac{66}{100} = 0.66$$

$$\ln(G_w) = \frac{\sum_{i=1}^{n} \varsigma_i \ln|d_i|}{\sum_{i=1}^{n} \varsigma_i} = \frac{-85.32896275}{100} = -0.8532896275$$

Geometric measure $= \exp(-0.8532896275) = 0.426011206 \approx 0.4260$

4.1.2.2 Binomial Sample

Consider a sample of size 100 from a Binomially distributed population with 30 trials and a probability of success 0.64. The frequency distribution and the estimation of average variation about the mean using geometric measure were as illustrated in Table 4.8;

Table 4.8: Application on weighted binomial sample

Number (v_i)	Frequency (ς_i)	$\varsigma_i v_i$	$d_i = v_i - \bar{v}$	$\lvert v_i - \bar{v}\rvert$	$\ln\lvert d_i\rvert = \ln\lvert v_i - \bar{v}\rvert$	$\varsigma_i \ln\lvert d_i\rvert = \varsigma_i \lvert v_i - \bar{v}\rvert$
11	2	22.0	-8.02	8.02	2.081938422	4.163876844
12	1	12.0	-7.02	7.02	1.948763218	1.948763218
13	1	13.0	-6.02	6.02	1.795087259	1.795087259

14	3	42.0	-5.02	5.02	1.613429934	4.840289801
15	4	60.0	-4.02	4.02	1.391281903	5.565127611
16	5	80.0	-3.02	3.02	1.105256831	5.526284157
17	13	221.0	-2.02	2.02	0.703097511	9.140267648
18	7	126.0	-1.02	1.02	0.019802627	0.138618391
19	21	399.0	-0.02	0.02	-3.912023005	-82.15248311
20	9	180.0	0.98	0.98	-0.020202707	-0.181824366
21	16	336.0	1.98	1.98	0.683096845	10.92954952
22	8	176.0	2.98	2.98	1.091923301	8.735386404
23	7	161.0	3.98	3.98	1.381281819	9.668972735
24	1	24.0	4.98	4.98	1.605429891	1.605429891
25	2	50.0	5.98	5.98	1.788420568	3.576841136
Total	100	1902.0				-14.69981287

$$Mean(\bar{v}) = \frac{\sum_{i=1}^{n} \varsigma_i v_i}{\sum_{i=1}^{n} \varsigma_i} = \frac{1902}{100} = 19.02$$

$$\ln(G_w) = \frac{\sum_{i=1}^{n} \varsigma_i \ln|d_i|}{\sum_{i=1}^{n} \varsigma_i} = \frac{-14.69981287}{100} = -0.1469981287$$

Geometric measure $= \exp(-0.1469981287) = 0.863295593 \approx 0.8633$

4.1.2.3 Geometric Sample

Consider a sample of size 100 from a Geometrically distributed population with a probability of success 0.5. The frequency distribution and the estimation of average variation about the mean using geometric measure were as illustrated in Table 4.9;

Table 4.9: Application on a weighted geometric dataset

Number (v_i)	Frequency (ς_i)	$\varsigma_i v_i$	$d_i = v_i - \bar{v}$	$\lvert v_i - \bar{v} \rvert$	$\ln\lvert d_i \rvert = \ln\lvert v_i - \bar{v} \rvert$	$\varsigma_i \ln\lvert d_i \rvert = \varsigma_i \lvert v_i - \bar{v} \rvert$
1	52	52	-1.12	1.12	0.113328685	5.893091636
2	22	44	-0.12	0.12	-2.120263536	-46.6457978
3	14	42	0.88	0.88	-0.127833372	-1.789667201
4	4	16	1.88	1.88	0.631271777	2.525087107

5	2	10	2.88	2.88	1.057790294	2.115580588
6	2	12	3.88	3.88	1.355835154	2.711670307
7	1	7	4.88	4.88	1.58514522	1.58514522
8	1	8	5.88	5.88	1.771556762	1.771556762
9	1	9	6.88	6.88	1.928618652	1.928618652
12	1	12	9.88	9.88	2.290512512	2.290512512
Total	100	212.0				-27.61420221

$$Mean(\bar{v}) = \frac{\sum_{i=1}^{n} \varsigma_i v_i}{\sum_{i=1}^{n} \varsigma_i} = \frac{212}{100} = 2.12$$

$$\ln(G_w) = \frac{\sum_{i=1}^{n} \varsigma_i \ln|d_i|}{\sum_{i=1}^{n} \varsigma_i} = \frac{-27.61420221}{100} = -0.2761420221$$

Geometric measure $= \exp(-0.2761420221) = 0.75870517 \approx 0.7587$

4.1.2.4 Chi-square Sample

Sample of size 100 from Chi-square distributed population with 1 degree of freedom was simulated, the frequency distribution and the estimation of average variation about the mean using geometric measure were as illustrated in Table 4.10;

Table 4.10: Application on weighted chi-square sample

Class	Midpoint (v)	Frequency (ς)	ςv	$\lvert d\rvert = \lvert v-\bar{v}\rvert$	$\ln\lvert d\rvert = \ln\lvert v-\bar{v}\rvert$	$\varsigma\ln\lvert d\rvert = \varsigma\ln\lvert v-\bar{v}\rvert$
0 – 2	1	82	82.0	0.5	-0.69314718	-56.83806881
2 – 4	3	12	36.0	1.5	0.40546511	4.865581297
4 – 6	5	5	25.0	3.5	1.25276297	6.263814842
6 – 8	7	1	7.0	5.5	1.70474809	1.704748092
Total		100	150.0			-44.00392457

$$Mean(\bar{v}) = \frac{\sum \varsigma v}{\sum \varsigma} = \frac{150}{100} = 1.5$$

$$\ln(G_w) = \frac{\sum \varsigma \ln|d|}{\sum \varsigma} = \frac{-44.00392457}{100} = -0.4400392457$$

Geometric measure $= \exp(-0.4400392457) = 0.644011146 \approx 0.6440$

4.1.2.5 Normal Distribution Sample

A sample of size 100 from a Normally distributed population with a mean of 10 and a standard deviation of 2 was simulated. The frequency distribution and the estimation of average variation about the mean using geometric measure were as illustrated in Table 4.11;

Table 4.11: Application on weighted normal distribution sample

Class	Midpoint (v)	Frequency (ς)	ςv	$d = v - \bar{v}$	$\lvert d\rvert = \lvert v - \bar{v}\rvert$	$\ln\lvert d\rvert = \ln\lvert v - \bar{v}\rvert$	$\varsigma \ln\lvert d\rvert = \varsigma \ln\lvert v - \bar{v}\rvert$
4 – 6	5	1	5.0	-5.3	5.3	1.6677	1.667706821
6 – 8	7	10	70.0	-3.3	3.3	1.1939	11.93922468
8 – 10	9	38	342.0	-1.3	1.3	0.2624	9.96984205
10 – 12	11	26	286.0	0.7	0.7	-0.3567	-9.273548542
12 – 14	13	24	312.0	2.7	2.7	0.9933	23.83804255
14 – 16	15	1	15.0	4.7	4.7	1.5476	1.547562509
Total		100	1030.0				39.68883007

$$Mean(\bar{v}) = \frac{\sum \varsigma v}{\sum \varsigma} = \frac{1030}{100} = 10.3$$

$$\ln(G_w) = \frac{\sum \varsigma \ln|d|}{\sum \varsigma} = \frac{39.68883007}{100} = 0.3968883007$$

Geometric measure $= \exp(0.3968883007) = 1.4871898033 \approx 1.4872$

4.1.2.6 F-distributed Sample

A sample of size 100 from an F-distributed population with 2 numerator and 5 denominator degrees of freedom was simulated. The frequency distribution and the estimation of average variation about the mean using geometric measure were as illustrated in Table 4.12

Table 4.12: Application on weighted F-distribution sample

Class	Midpoint (v)	Frequency (ς)	ςv	$\lvert d\rvert = \lvert v-\bar{v}\rvert$	$\ln\lvert d\rvert = \ln\lvert v-\bar{v}\rvert$	$\varsigma \ln\lvert d\rvert = \varsigma \ln\lvert v-\bar{v}\rvert$
0 – 2	1	69	69.0	1.055	0.05354077	3.694312918
2 – 4	3	18	54.0	0.945	-0.05657035	-1.018266327
4 – 6	5	7	35.0	2.945	1.08010882	7.56076172
6 – 8	7	5	35.0	4.945	1.59837697	7.991884825
8 – 10	9	0	0.0	6.945	1.93802198	0
10 – 15	12.5	1	12.5	10.445	2.34612340	2.346123395
	Total	100	205.5			20.57481653

$$Mean(\bar{v}) = \frac{\sum \varsigma v}{\sum \varsigma} = \frac{205.5}{100} = 2.055$$

$$\ln(G_w) = \frac{\sum \varsigma \ln|d|}{\sum \varsigma} = \frac{20.57481653}{100} = 0.2057481653$$

Geometric measure $= \exp(0.0.2057481653) = 1.2284438 \approx 1.2284$

Based on the illustration shown by the calculations, the geometric measure was able to be used to estimate variation about the mean for selected number of weighted discrete and continuous datasets.

4.1.3 Application on Probability Mass Functions

The study considered various probability mass functions to illustrate how the function can be used in estimating the average variation from the mean for probability mass functions. The results were as illustrated in 4.2.3.1 to 4.2.3.6;

4.1.3.1 Coin Tossing Gambling Game

In a coin-tossing gambling game, a player loses 1 shilling whenever a coin tosses head and gains 2 shillings whenever the coin tosses tail. The task was to estimate the average variation of the gain using a geometric measure of variation from the mean.

The calculation of the estimate was as illustrated in table 4.13

Table 4.13: Application on Coin Tossing Gambling Game

Gain (v_i)	Probability $\varsigma(v_i)$	$v_i\varsigma(v_i)$	d_i	$\lvert d_i\rvert$	$\ln\lvert d_i\rvert$	$\varsigma(v_i)\ln\lvert d_i\rvert$
-1	0.5	-0.5	-1.5	1.5	0.405465108	0.202732554
2	0.5	1	1.5	1.5	0.405465108	0.202732554
Total	1	0.5				0.405465108

$G_{pm} = \exp(0.405465108) = 1.5$

4.1.3.2 Dice

When an individual rolls a die, they can get one of the numbers ranging from 1 to 6. Estimation of the average variation in obtaining a number the die is rolled was as illustrated in Table 4.14;

Table 4.14: Application on Rolling of Dice

Number (v_i)	Probability $\varsigma(v_i)$	$v_i\varsigma(v_i)$	d_i	$\lvert d_i\rvert$	$\ln\lvert d_i\rvert$	$\varsigma(v_i)\ln\lvert d_i\rvert$
1	0.166666667	0.166666667	-2.5	2.5	0.916290732	0.152715122
2	0.166666667	0.333333333	-1.5	1.5	0.405465108	0.067577518
3	0.166666667	0.5	-0.5	0.5	-0.693147181	-0.11552453
4	0.166666667	0.666666667	0.5	0.5	-0.693147181	-0.11552453
5	0.166666667	0.833333333	1.5	1.5	0.405465108	0.067577518
6	0.166666667	1	2.5	2.5	0.916290732	0.152715122
Total		3.5				0.20953622

$G_{pm} = \exp(0.20953622) = 1.2331060037 \approx 1.2331$

4.1.3.3 Bernoulli Distribution

Consider a Bernoulli distribution with probability mass function

$$\varsigma(V) = \begin{cases} \varsigma^{v_i}(1-\varsigma)^{1-v_i} & V = 0,1 \\ 0 & Otherwise \end{cases} \tag{4.1}$$

Where ς is the probability of success, the natural logarithm of the geometric measure of variation about the mean can be estimated for the function by

$$\ln\left(G_{pm}\right)=\sum_{i=0}^{1}\varsigma^{v_i}\left(1-\varsigma\right)^{1-v_i}\ln\left|v_i-\varsigma\right| \tag{4.2}$$

Hence, the geometric measure of variation about the mean would be given by

$$G_{pm}=\begin{cases}\exp\left(\sum_{i=0}^{1}\varsigma^{v_i}\left(1-\varsigma\right)^{1-v_i}\ln\left|v_i-\varsigma\right|\right) & v_0\neq v_1\neq\varsigma\\ 0 & v_0=v_1=\varsigma\end{cases} \tag{4.3}$$

4.1.3.4 Binomial Distribution

Consider a binomial distribution with probability mass function

$$\varsigma(V)=\begin{cases}\begin{pmatrix}t\\ v_i\end{pmatrix}\varsigma^{v_i}\left(1-\varsigma\right)^{t-v_i} & V=0,1,2,\ldots,t\\ 0 & Otherwise\end{cases} \tag{4.4}$$

Where ς is the probability of success, the natural logarithm of the geometric measure of variation about the mean can be estimated for the function by

$$\ln\left(G_{pm}\right)=\sum_{i=0}^{t}\begin{pmatrix}t\\ v_i\end{pmatrix}\varsigma^{v_i}\left(1-\varsigma\right)^{t-v_i}\ln\left|v_i-t\varsigma\right| \tag{4.5}$$

Hence, the geometric measure of variation about the mean would be given by

$$G_{pm}=\begin{cases}\exp\left(\sum_{i=0}^{t}\begin{pmatrix}t\\ v_i\end{pmatrix}\varsigma^{v_i}\left(1-\varsigma\right)^{1-v_i}\ln\left|v_i-t\varsigma\right|\right) & v_0\neq v_1\neq v_2\neq\ldots v_t\neq t\varsigma\\ 0 & v_0=v_1=v_2=\ldots v_t=t\varsigma\end{cases} \tag{4.6}$$

4.1.3.5 Geometric Distribution

Consider a geometric distribution with probability mass function

$$\varsigma(V)=\begin{cases}\varsigma\left(1-\varsigma\right)^{v_i} & V=0,1,2,\ldots\\ 0 & Otherwise\end{cases} \tag{4.7}$$

Where ς is the probability of success, the natural logarithm of the geometric measure of variation about the mean can be estimated for the function by

$$\ln\left(G_{pm}\right)=\sum_{i=0}^{\infty}\varsigma(1-\varsigma)^{v_i}\ \ln\left|v_i-\frac{1-\varsigma}{\varsigma}\right| \tag{4.8}$$

$$\ln\left(G_{pm}\right)=\sum_{i=0}^{\infty}\varsigma(1-\varsigma)^{v_i}\ \ln\left|\frac{\varsigma v_i-1+\varsigma}{\varsigma}\right| \tag{4.9}$$

$$\ln\left(G_{pm}\right)=\sum_{i=0}^{\infty}\varsigma(1-\varsigma)^{v_i}\ \ln\left|\frac{\varsigma v_i+\varsigma-1}{\varsigma}\right| \tag{4.10}$$

$$\ln\left(G_{pm}\right)=\sum_{i=0}^{\infty}\varsigma(1-\varsigma)^{v_i}\ \ln\left|\frac{\varsigma(v_i+1)-1}{\varsigma}\right| \tag{4.11}$$

Hence, the geometric measure of variation about the mean would be given by

$$G_{pm}=\begin{cases}\exp\left(\sum_{i=0}^{\infty}\varsigma(1-\varsigma)^{v_i}\ \ln\left|\frac{\varsigma(v_i+1)-1}{\varsigma}\right|\right) & v_0\neq v_1\neq v_2\neq\ldots\neq v_\infty\neq\frac{1-\varsigma}{\varsigma}\\ 0 & v_0=v_1=v_2=\ldots=v_\infty=\frac{1-\varsigma}{\varsigma}\end{cases} \tag{4.12}$$

4.1.3.6 Poisson Distribution

Consider a Poisson distribution with probability mass function

$$\varsigma(V)=\begin{cases}\frac{e^{-\varsigma}\varsigma^{v_i}}{v_i!} & V=0,1,2,3,\ldots\\ 0 & Otherwise\end{cases} \tag{4.12}$$

Where ς is the average rate of event occurrence, the natural logarithm of the geometric measure of variation about the mean can be estimated for the function by;

$$\ln(G_{pm}) = \sum_{i=0}^{\infty} \frac{e^{-\varsigma}\varsigma^{v_i}}{v_i!} \ln|v_i - \varsigma| \tag{4.13}$$

Hence, the geometric measure of variation about the mean would be given by

$$G_{pm} = \begin{cases} \exp\left(\sum_{i=0}^{\infty} \frac{e^{-\varsigma}\varsigma^{v_i}}{v_i!} \ln|v_i - \varsigma| \right) & v_0 \neq v_1 \neq v_2 \neq \dots v_\infty \neq \varsigma \\ 0 & v_0 = v_1 = v_2 = \dots v_\infty = \varsigma \end{cases} \tag{4.14}$$

The above illustrations show how the geometric measure about the mean can be used to estimate the average variations about the mean of probability mass functions.

4.1.4 Application on Probability Density Functions

The study considered various probability density functions to illustrate how the function can be used in estimating the average variation from the mean for probability density functions. The results were as illustrated in 4.2.4.1 to 4.2.4.4;

4.1.4.1 Simple Probability Density Function

Consider a random v variable which is distributed in the interval $[0 \quad 1]$ with a probability density function;

$$\varsigma(v) = \begin{cases} 1 & 0 \leq v \leq 1 \\ 0 & elsewhere \end{cases}$$

We can estimate the average variation about the mean for v using the geometric measure as follows;

The geometric average deviation from the mean is given by the function;

$$G_{pd} = \begin{cases} \exp\left(\int_a^b \varsigma(v) \ln|v - \bar{v}|.dv \right) & v \neq \bar{v} \\ 0 & v = \bar{v} \end{cases}$$

Therefore, for the function, we can obtain the expected value of the distribution as follows;

$$E(V) = \int_0^1 v.dv = \frac{1}{2}v^2 + c\Big|_0^1 = \left[\frac{1}{2} - 0\right] = \frac{1}{2}$$

The natural logarithm of the geometric measure of variation from the mean by definition is given by the formula;

$$\ln(G_{pd}) = E(\ln|v - \bar{v}|) = \int_a^b \varsigma(v) \ln|v - \bar{v}|.dv$$

Hence, for the above probability density function

$$\ln(G_{pd}) = E\left(\ln\left|v - \frac{1}{2}\right| \right) = \int_0^1 1 \ln|v - \bar{v}|.dv$$
$$= 1\int_0^1 \ln\left|v - \frac{1}{2}\right|.dv = 1\left[\left(v - \frac{1}{2}\right)\ln\left|v - \frac{1}{2}\right| - v + \frac{1}{2}\right]_0^1$$
$$= 1\left[\frac{1}{2}\ln\left(\frac{1}{2}\right) - 1 + \frac{1}{2}\right] - 1\left[-\frac{1}{2}\ln\left(\frac{1}{2}\right) + \frac{1}{2}\right]$$
$$= [-0.84657] - [0.84657]$$
$$= -1.69315$$

Therefore, the geometric measure of variation from the mean will be given by;

$$G_{pd} = \exp(-1.69315) = 0.18394$$

4.1.4.2 Uniform Distribution

Consider a random variable v which is distributed uniformly in the interval $[2 \quad 10]$. By definition, the probability density function of v will be given by

$$\varsigma(v)=\begin{cases}\dfrac{1}{10-2} & 2\le v\le 10\\ 0 & otherwise\end{cases}$$

The expectation of the function is given by

$$E(v)=\frac{10+2}{2}=6$$

Therefore, the natural logarithm of geometric measure of variation from the mean for the distribution will be given by

$$\ln(G_{pd})=E(\ln|v-6|)=\int_{2}^{10}\frac{1}{10-2}\ln|v-6|.dv$$
$$=\frac{1}{8}\int_{2}^{10}\ln|v-6|.dv=\frac{1}{8}\left[(v-6)\ln|v-6|-v+6\right]_{2}^{10}$$
$$=\frac{1}{8}\left[(4\ln(4)-10+6)-(-4\ln(4)-2+6)\right]$$
$$=\frac{1}{8}\left[1.54518+1.54518\right]=0.386294$$

Therefore, the geometric measure of variation from the mean will be given by

$$G_{pd}=\exp(0.386294)=1.471518\approx 1.4715$$

4.1.4.3 Exponential Probability Density Function

Consider an exponential distribution with probability density function

$$\varsigma(v)=\begin{cases}\varsigma e^{-\varsigma v} & v\geq 0\\ 0 & otherwise\end{cases}$$

By definition of expectation of an exponential function

$$E(v)=\frac{1}{\varsigma}$$

Therefore, the natural logarithm of geometric measure about the mean is given by;

$$\ln\left(G_{pd}\right)=E\left(\ln\left|v-\frac{1}{\varsigma}\right|\right)=\int_{0}^{\infty}\varsigma e^{-\varsigma v}\ln\left|v-\frac{1}{\varsigma}\right|.dv$$

Hence, the geometric measure of variation about the mean would be given by

$$G_{pd}=\begin{cases}\exp\left(\int_{0}^{\infty}\varsigma e^{-\varsigma v}\ln\left|v-\frac{1}{\varsigma}\right|.dv\right) & v\neq\frac{1}{\varsigma}\\ 0 & v=\frac{1}{\varsigma}\end{cases}$$

The above illustrations show that the geometric measure of variation from the mean can be used to estimate the average deviation from the mean. However, the function faces a challenge in the estimation especially when the limits are infinite. Also, the challenge arises when conducting partial integration for complex probability density functions that do not have finite integrals for the integration by parts. For example, consider the case of standard normal probability density function;

4.1.4.4 Standard Normal

Consider a standard normal distribution for a random variable v illustrated as follows

$$\varsigma(v)=\begin{cases}\frac{1}{\sqrt{2\pi}}e^{-\frac{v^2}{2}} & -\infty\le v\le\infty\\ 0 & otherwise\end{cases}$$

Therefore, the natural logarithm of geometric measure about the mean is given by;

$$\ln(G_{pd})=E(\ln|v|)=\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}e^{-\frac{v^2}{2}}\ln|v|.dv$$

Hence, the geometric measure of variation about the mean would be given by

$$G_{pd}=\begin{cases}\exp\left(\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}e^{-\frac{v^2}{2}}\ln|v|.dv\right) & v\neq 0\\ 0 & v=0\end{cases}$$

This shows that it is almost impossible to integrate such functions. This is a shortcoming of the geometric measure of variation especially during its application on probability density functions.

4.2 Test for Difference and Efficiency Compared to Standard Deviation

A paired sample t-test was used to test for a significant difference in the sample estimates obtained by the geometric measure in comparison to the estimates obtained by the standard deviation. The test was also used to check if the geometric measure estimates were significantly smaller than those of standard deviation. In order to assess the efficiency of the geometric measure about the mean, three measures of efficiency were used to compare the efficiency of the measure in comparison to standard deviation (Mean Squared Error, Relative Efficiency, and Bias). The study used both small samples of size 10 each and large samples of size 200 each to assess the efficiency. The results were as illustrated below.

4.2.1 Small Binary Samples

Twenty Bernoulli samples each of size 10 with 0.7 probability of success were simulated. The estimates for the geometric measure of variation about the mean and standard deviation for each of the samples were as illustrated in Table 4.15;

Table 4.15: Small Binary Sample Estimates

Sample Number	**Standard Deviation**	**Geometric Measure of Variation**
1	0.483	0.387
2	0.516	0.470
3	0.516	0.470
4	0.527	0.500
5	0.422	0.264
6	0.422	0.264
7	0.422	0.264
8	0.316	0.125
9	0.422	0.264
10	0.516	0.470
11	0.316	0.125
12	0.483	0.387
13	0.422	0.264
14	0.516	0.470
15	0.422	0.264
`16	0.316	0.125
17	0.527	0.500
18	0.483	0.387
19	0.316	0.125
20	0.516	0.470

The test for significant difference between the standard deviation measures and the geometric measure of variation showed that there was a significant difference between the geometric measures estimates and the standard deviation estimates (t=8.145, df=19, p-value<0.001). The test showed that the average estimates for the geometric measure of variation from the mean were significantly smaller than those of standard deviation (p-value<0.001).

The test results for the efficiency of the geometric measure in comparison to standard deviation, based on the four measures of efficiency were as illustrated in Table 4.16;

Table 4.16: Efficiency Measures for Small Binary Sample

Measure of Efficiency	Standard Deviation	Geometric Measure of Variation
Bias	-0.013	-0.052
Mean Squared Error	0.006	0.021
Relative Efficiency	0.2857	

The results in table 4.16 illustrate that standard deviations had a smaller bias compared to geometric measure. It also had a smaller Mean Squared Error compared to geometric measure. Comparing the MSE of the two measures through relative efficiency, it is observed that the relative efficiency was less than 1 meaning that standard deviation was more efficient than the geometric measure in estimating the average variation about the mean for small binary samples.

4.2.2 Large Binary Samples

Twenty Bernoulli samples each of size 200 with 0.7 probability of success were simulated. The estimates for the geometric measure of variation about the mean and standard deviation for each of the samples were as illustrated in Table 4.17;

Table 4.17: Large Binary Sample Estimates

Sample Number	Standard Deviation	Geometric Measure of Variation
1	0.434	0.329
2	0.455	0.376
3	0.462	0.392
4	0.445	0.353
5	0.466	0.402
6	0.434	0.329
7	0.448	0.359
8	0.448	0.359
9	0.448	0.359

10	0.442	0.347
11	0.455	0.376
12	0.475	0.426
13	0.475	0.426
14	0.473	0.422
15	0.471	0.417
`16	0.455	0.376
17	0.448	0.359
18	0.483	0.447
19	0.445	0.353
20	0.468	0.407

The test for significant difference between the standard deviation measures and the geometric measure of variation showed that there was a significant difference between the geometric measures estimates and the standard deviation estimates ($t=16.6956$, $df=19$, p-value<0.001). The test showed that the average estimates for the geometric measure of variation from the mean were significantly smaller than those of standard deviation (p-value<0.001).

The test results for the efficiency of the geometric measure in comparison to standard deviation, based on the four measures of efficiency were as illustrated in Table 4.18;

Table 4.18: Efficiency Measures for Large Binary Sample

Measure of Efficiency	**Standard Deviation**	**Geometric Measure of Variation**
Bias	-0.0003	-0.0021
Mean Squared Error	0.0002	0.0011
Relative Efficiency	0.1818	

The results in Table 4.18 illustrate that standard deviations had a smaller bias compared to geometric measure. It also had a smaller Mean Squared Error compared to geometric measure. Comparing the MSE of the two measures through relative efficiency, it is observed that the relative efficiency was less than 1 meaning that standard deviation was more efficient than the geometric measure in estimating the average variation about the mean for large binary samples.

4.2.3 Small Geometric Sample

Twenty Geometric distributed samples each of size 20 with 0.5 probability of success were simulated. The estimates for the geometric measure of variation about the mean and standard deviation for each of the samples were as illustrated in Table 4.19;

Table 4.19: Estimates for Small Geometric Sample

Sample Number	**Standard Deviation**	**Geometric Measure of Variation**
1	0.632	0.363
2	2.003	1.147
3	1.729	1.189
4	3.651	1.661
5	0.876	0.494
6	1.506	0.819
7	2.573	1.075
8	1.080	0.776
9	0.823	0.614
10	1.287	0.873
11	1.449	0.777
12	0.527	0.500
13	1.776	1.293
14	0.699	0.555
15	1.075	0.720
`16	1.229	0.839
17	1.491	1.196
18	1.197	0.339
19	0.823	0.614
20	1.054	1.000

The test for significant difference between the standard deviation measures and the geometric measure of variation showed that there was a significant difference between the geometric measures estimates and standard deviation estimates (t=4.9274, df=19, p-value<0.001). The test showed that the average estimates for the geometric measure of variation from the mean were significantly smaller than those of standard deviation (p-value<0.001).

The test results for the efficiency of the geometric measure in comparison to standard deviation, based on the four measures of efficiency were as illustrated in table 4.20;

Table 4.20: Efficiency Estimates for Small Geometric Sample

Measure of Efficiency	**Standard Deviation**	**Geometric Measure of Variation**
Bias	0.186	0.087
Mean Squared Error	0.548	0.119
Relative Efficiency	4.605	

The results in Table 4.20 illustrated that geometric measure had a smaller bias compared to standard deviation. It also had a smaller Mean Squared Error compared to standard deviation. Comparing the MSE of the two measures through relative efficiency; it is observed that the relative efficiency was more than 1 meaning that geometric measure was more efficient than the standard deviation in estimating the average variation about the mean for small geometric samples.

4.2.4 Large Geometric Discrete Sample

Twenty Geometric distributed samples each of size 200 with 0.5 probability of success were simulated. The estimates for the geometric measure of variation about the mean and standard deviation for each of the samples were as illustrated in Table 4.21;

Table 4.21: Estimates for Large Geometric Sample

Sample Number	**Standard Deviation**	**Geometric Measure of Variation**
1	1.293	0.694
2	1.522	0.631
3	1.268	0.462
4	1.252	0.415
5	1.497	0.479

6	1.433	0.558
7	1.422	0.594
8	1.326	0.489
9	1.447	0.465
10	1.405	0.256
11	1.463	0.462
12	1.423	0.448
13	1.146	0.710
14	1.195	0.538
15	1.162	0.676
`16	1.538	0.500
17	1.423	0.595
18	1.321	0.606
19	1.662	0.536
20	1.342	0.506

The test for significant difference between the standard deviation measures and the geometric measure of variation showed that there was a significant difference between the geometric measures estimates and the standard deviation estimates (t=19.5459, df=19, p-value<0.001). The test showed that the average estimates for the geometric measure of variation from the mean were significantly smaller than those of standard deviation (p-value<0.001).

The test results for the efficiency of the geometric measure in comparison to standard deviation, based on the four measures of efficiency were as illustrated in Table 4.22;

Table 4.22: Efficiency Measures for Large Geometric Sample

Measure of Efficiency	**Standard Deviation**	**Geometric Measure of Variation**
Bias	-0.006	0.042
Mean Squared Error	0.017	0.013
Relative Efficiency	1.308	

Checking at the results in Table 4.22, based on bias, standard deviation had a smaller bias compared to geometric measure. However, checking at the MSE of the two measures, the geometric measure had a smaller MSE compared to standard deviation. As a result, comparing the relative efficiency of the two measures, the geometric measure was considered to be more efficient than standard deviation because the relative efficiency was more than 1. This shows that geometric measure was more efficient in estimating the average variation about the mean for large geometric samples.

4.2.5 Small Countable Sample

Poisson distribution is one of the known countable discrete distributions, in order to assess the efficiency of the geometric measure of variation in estimating the average variation about the mean for small countable discrete distributed samples, twenty samples each of size 10 from a Poisson distribution with a mean of 50 was assessed. The results of the geometric measure of variation from the mean and standard deviation for the 20 samples were as illustrated in Table 4.23;

Table 4.23: Estimates for Small Countable Sample

Sample Number	Standard Deviation	Geometric Measure of Variation
1	5.794	2.983
2	5.797	3.011
3	6.851	3.831
4	6.746	4.543
5	4.122	2.708
6	10.840	6.806
7	5.567	2.711
8	9.914	6.302
9	9.686	4.506
10	6.222	3.498
11	8.206	3.588
12	8.436	5.153
13	6.502	3.258
14	8.779	6.363
15	5.432	2.090
`16	7.391	4.866
17	6.346	4.700
18	7.557	4.141

19	6.546	2.076
20	7.005	3.383

The test for significant difference between the standard deviation measures and the geometric measure of variation showed that there was a significant difference between the geometric measures estimates and the standard deviation estimates (t=14.943, df=19, p-value<0.001). The test showed that the average estimates for the geometric measure of variation from the mean were significantly smaller than those of standard deviation (p-value<0.001).

The test results for the efficiency of the geometric measure in comparison to standard deviation, based on the four measures of efficiency were as illustrated in table 4.24;

Table 4.24: Efficiency Measures for Small Countable Samples

Measure of Efficiency	**Standard Deviation**	**Geometric Measure of Variation**
Bias	0.133	0.186
Mean Squared Error	2.726	1.833
Relative Variation	1.4872	

Checking at the results in table 4.24, based on bias, standard deviation had a smaller bias compared to geometric measure. However, checking at the MSE of the two measures, the geometric measure had a smaller MSE compared to standard deviation. As a result, comparing the relative efficiency of the two measures, the geometric measure was considered to be more efficient than standard deviation because the relative efficiency was more than 1. This shows that geometric measure was more efficient in estimating the average variation about the mean for large geometric samples.

4.2.6 Large Countable Sample

Considering large samples of discrete countable random variables, twenty samples each of size 200 from a Poisson distribution with a mean of 50 were assessed. The results of the geometric measure of variation from the mean and standard deviation for the 20 samples were as illustrated in Table 4.25;

Table 4.25: Estimates for Large Countable Samples

Sample Number	Standard Deviation	Geometric Measure of Variation
1	7.215	3.403
2	6.434	3.598
3	7.100	4.197
4	6.935	2.835
5	7.178	3.953
6	7.425	3.978
7	7.182	3.573
8	7.154	3.783
9	6.959	3.638
10	7.212	3.794
11	7.461	4.151
12	6.831	4.016
13	6.862	4.007
14	6.513	3.237
15	7.519	3.561
`16	7.181	3.501
17	7.188	3.721
18	7.422	4.406
19	7.593	4.238
20	7.161	4.052

The test for significant difference between the standard deviation measures and the geometric measure of variation showed that there was a significant difference between the geometric measures estimates and the standard deviation estimates (t=40.9367, df=19, p-value<0.001). The test showed that the average estimates for the geometric measure of variation from the mean were significantly smaller than those of standard deviation (p-value<0.001).

The test results for the efficiency of the geometric measure in comparison to standard deviation, based on the four measures of efficiency were as illustrated in table 4.26;

Table 4.26: Efficiency Measures for Large Countable Samples

Measure of Efficiency	Standard Deviation	Geometric Measure of Variation
Bias	-0.005	0.066
Mean Squared Error	0.088	0.140
Relative Efficiency	0.6286	

The results in table 4.26 illustrated that standard deviations had a smaller bias compared to geometric measure. It also had a smaller Mean Squared Error compared to geometric measure. Comparing the MSE of the two measures through relative efficiency; it is observed that the relative efficiency was less than 1 meaning that standard deviation was more efficient than the geometric measure in estimating the average variation about the mean for large countable samples.

4.2.7 Small Normal Distributed Samples

Past studies have shown that standard deviation always assumes normality when estimating the average variation from the mean. In order to assess the efficiency of the geometric measures of variation in comparison to the standard deviation in estimating the variation from the mean for small normally distributed samples, twenty samples each of size 10 from a normal population with a mean of 18 and standard deviation of 2 were simulated and assessed. The results of standard deviation and the geometric measure of variation from the mean for the samples were as illustrated in table 4.27;

Table 4.27: Estimates for Small Normal Samples

Sample Number	Standard Deviation	Geometric Measure of Variation
1	0.960	0.351
2	0.892	0.461
3	1.161	0.715
4	1.559	0.820
5	1.521	0.696
6	1.689	0.846

7	1.210	0.731
8	1.472	0.987
9	1.615	1.104
10	1.660	0.930
11	0.736	0.332
12	1.264	0.755
13	1.080	0.594
14	1.477	0.691
15	1.105	0.768
`16	1.328	0.624
17	1.258	0.718
18	1.898	1.399
19	1.521	0.662
20	1.637	0.933

The test for significant difference between the standard deviation measures and the geometric measure of variation showed that there was a significant difference between the geometric measures estimates and the standard deviation estimates (t=16.439, df=19, p-value<0.001). The test showed that the average estimates for the geometric measure of variation from the mean were significantly smaller than those of standard deviation (p-value<0.001).

The test results for the efficiency of the geometric measure in comparison to standard deviation, based on the four measures of efficiency were as illustrated in table 4.28;

Table 4.28: Efficiency Measures for Small Normal Samples

Measure of Efficiency	**Standard Deviation**	**Geometric Measure of Variation**
Bias	0.106	0.004
Mean Squared Error	0.098	0.058
Relative Efficiency	1.6897	

The results in Table 4.28 illustrated that geometric measure had a smaller bias compared to standard deviation. It also had a smaller Mean Squared Error compared to standard deviation.

Comparing the MSE of the two measures through relative efficiency; it is observed that the relative efficiency was more than 1, meaning that geometric measure was more efficient than the standard deviation in estimating the average variation about the mean for small Normal samples.

4.2.8 Large Normal Distributed Samples

Considering large samples of normally distributed datasets, twenty samples each of size 200 from a normal population with a mean of 18 and a standard deviation of 2 were simulated and assessed. The results of standard deviation and the geometric measure of variation from the mean for the samples were as illustrated in table 4.29;

Table 4.29: Estimates for Large Normal Samples

Sample Number	Standard Deviation	Geometric Measure of Variation
1	1.539	0.858
2	1.408	0.758
3	1.540	0.939
4	1.571	0.912
5	1.410	0.763
6	1.556	0.853
7	1.694	0.878
8	1.489	0.789
9	1.466	0.720
10	1.397	0.777
11	1.387	0.804
12	1.606	0.891
13	1.473	0.805
14	1.416	0.800
15	1.483	0.814
`16	1.399	0.781
17	1.404	0.759
18	1.444	0.771
19	1.478	0.788
20	1.544	0.861

The test for significant difference between the standard deviation measures and the geometric measure of variation showed that there was a significant difference between the geometric measures estimates and the standard deviation estimates (t=56.119, df=19, p-value<0.001). The test showed that the average estimates for the geometric measure of variation from the mean were significantly smaller than those of standard deviation (p-value<0.001).

The test results for the efficiency of the geometric measure in comparison to standard deviation, based on the four measures of efficiency were as illustrated in table 4.30;

Table 4.30: Efficiency Measures for Large Normal Samples

Measure of Efficiency	**Standard Deviation**	**Geometric Measure of Variation**
Bias	0.001	0.002
Mean Squared Error	0.007	0.003
Relative Efficiency	2.333	

Checking at the results in Table 4.30, based on bias, standard deviation had a smaller bias compared to geometric measure. However, checking at the MSE of the two measures, the geometric measure had a smaller MSE compared to standard deviation. As a result, comparing the relative efficiency of the two measures, the geometric measure was considered to be more efficient than standard deviation because the relative efficiency was more than 1. This shows that geometric measure was more efficient in estimating the average variation about the mean for large normal samples.

4.2.9 Small Skewed Samples

Past studies have shown that standard deviation is affected by skewed datasets. As a result, if a new measure of variation is to be developed, then it should be one that is not affected by the skewness of data. In order to assess the efficiency of the geometric measures of variation in comparison to standard deviation in estimating the variation from the mean for small skewed samples, twenty samples, each of size 10 from a chi-square distribution with 1 degree of freedom were simulated and assessed. The results of standard deviation and the geometric measure of variation from the mean for the samples were as illustrated in table 4.31;

Table 4.31: Estimates for small skewed samples

Sample Number	Standard Deviation	Geometric measure of variation
1	1.102	0.482
2	1.242	0.587
3	1.718	0.899
4	0.551	0.323
5	1.127	0.554
6	1.724	0.887
7	2.342	1.463
8	1.394	0.769
9	1.532	0.910
10	1.136	0.508
11	3.244	1.653
12	1.325	0.340
13	1.558	0.741
14	1.380	1.049
15	0.815	0.572
`16	0.700	0.562
17	1.156	0.865
18	1.274	0.671
19	0.285	0.219
20	1.083	0.571

The test for significant difference between the standard deviation measures and the geometric measure of variation showed that there was a significant difference between the geometric measures estimates and the standard deviation estimates (t=7.744, df=19, p-value<0.001). The test showed that the average estimates for the geometric measure of variation from the mean were significantly smaller than those of standard deviation (p-value<0.001).

The test results for the efficiency of the geometric measure in comparison to standard deviation, based on the four measures of efficiency were as illustrated in Table 4.32;

Table 4.32: Efficiency Measures for small Skewed samples

Measure of Efficiency	**Standard Deviation**	**Geometric Measure of Variation**
Bias	0.149	0.006
Mean Squared Error	0.406	0.121
Relative Efficiency	3.355	

The results in table 4.32 illustrated that geometric measure had a smaller bias compared to standard deviation. It also had a smaller Mean Squared Error compared to standard deviation. Comparing the MSE of the two measures through relative efficiency, it is observed that the relative efficiency was more than 1, meaning that geometric measure was more efficient than the standard deviation in estimating the average variation about the mean for small skewed samples.

4.2.10 Large Skewed Samples

Considering large continuous skewed samples, the assessment on the efficiency of geometric measure of variation from the mean in estimating the average deviation from the mean in comparison to standard deviation, twenty samples, each of size 200 from a chi-square distribution with 1 degree of freedom were simulated and assessed. The results of standard deviation and the geometric measure of variation from the mean for the samples were as illustrated in table 4.33;

Table 4.33: Estimates for Large Skewed Sample

Sample Number	**Standard Deviation**	**Geometric Measure of Variation**
1	1.548	0.605
2	1.330	0.611
3	1.196	0.559
4	1.360	0.608
5	1.415	0.677
6	1.298	0.578

7	1.333	0.632
8	1.442	0.627
9	1.350	0.664
10	1.855	0.757
11	1.249	0.629
12	1.265	0.627
13	1.248	0.632
14	1.527	0.706
15	1.313	0.537
`16	1.260	0.554
17	1.798	0.724
18	1.314	0.621
19	1.338	0.498
20	1.414	0.670

The test for significant difference between the standard deviation measures and the geometric measure of variation showed that there was a significant difference between the geometric measures estimates and the standard deviation estimates ($t=25.292$, $df=19$, $p\text{-value}<0.001$). The test showed that the average estimates for the geometric measure of variation from the mean were significantly smaller than those of standard deviation ($p\text{-value}<0.001$).

The test results for the efficiency of the geometric measure in comparison to standard deviation, based on the four measures of efficiency were as illustrated in table 4.34;

Table 4.34: Efficiency Estimates for Large Skewed Samples

Measure of Efficiency	**Standard Deviation**	**Geometric Measure of Variation**
Bias	0.010	0.003
Mean Squared Error	7.844	1.578
Relative Efficiency	4.971	

The results in Table 4.34 illustrated that geometric measure had a smaller bias compared to standard deviation. It also had a smaller Mean Squared Error compared to standard deviation.

Comparing the MSE of the two measures through relative efficiency; it is observed that the relative efficiency was more than 1, meaning that geometric measure was more efficient than the standard deviation in estimating the average variation about the mean for large skewed samples.

4.2.11 Small Peaked Samples

Most peaked datasets are known to be prone to outliers because most of the observations are clustered in the middle with very few observations being observed at the extremes. Past studies have also determined that standard deviation is always affected outliers in the datasets. Hence a new measure of variation should not be affected by outliers. In order to assess the efficiency of the geometric measures of variation in comparison to standard deviation in estimating the variation from the mean for small peaked samples, twenty samples, each of size 10 from a t-distribution with 1 degree of freedom were simulated and assessed. The results of standard deviation and the geometric measure of variation from the mean for the samples were as illustrated in table 4.35;

Table 4.35: Estimates for Small Peaked Sample

Sample Number	Standard Deviation	Geometric Measure of Variation
1	5.737	1.779
2	8.483	2.535
3	3.614	1.483
4	5.542	2.053
5	3.598	1.388
6	3.450	1.842
7	4.584	0.976
8	6.904	1.264
9	1.625	1.020
10	4.758	1.051
11	10.302	2.842
12	1.478	0.926
13	8.990	4.707
14	1.856	0.653
15	2.922	1.403
`16	0.591	0.376

17	6.291	2.104
18	5.593	1.802
19	1.480	0.311
20	1.247	0.599

The test for significant difference between the standard deviation measures and the geometric measure of variation showed that there was a significant difference between the geometric measures estimates and the standard deviation estimates (t=6.387, df=19, p-value<0.001). The test showed that the average estimates for the geometric measure of variation from the mean were significantly smaller than those of standard deviation (p-value<0.001).

The test results for the efficiency of the geometric measure in comparison to standard deviation, based on the four measures of efficiency were as illustrated in table 4.36;

Table 4.36: Efficiency Measures for Small Peaked Samples

Measure of Efficiency	**Standard Deviation**	**Geometric Measure of Variation**
Bias	0.659	0.595
Mean Squared Error	7.788	1.323
Relative Efficiency	5.887	

The results in Table 4.36 illustrated that geometric measure had a smaller bias compared to standard deviation. It also had a smaller Mean Squared Error compared to standard deviation. Comparing the MSE of the two measures through relative efficiency, it is observed that the relative efficiency was more than 1, meaning that geometric measure was more efficient than the standard deviation in estimating the average variation about the mean for small peaked samples.

4.2.12 Large Peaked Continuous Samples

Considering large continuous peaked samples, the assessment on the efficiency of geometric measure of variation from the mean in estimating the average deviation from the mean in comparison to standard deviation, twenty samples, each of size 200 from a t-distribution with 1 degree of freedom were simulated and assessed. The results of standard deviation and the geometric measure of variation from the mean for the samples were as illustrated in Table 4.37;

Table 4.37: Estimates for Large Peaked Samples

Sample Number	Standard Deviation	Geometric Measure of Variation
1	19.847	2.127
2	15.192	1.558
3	6.501	0.869
4	69.818	4.560
5	8.195	0.957
6	7.682	0.987
7	20.534	0.892
8	7.972	1.118
9	36.581	3.241
10	23.838	1.784
11	20.142	1.539
12	13.341	1.036
13	227.637	13.728
14	4.590	1.109
15	8.292	0.982
`16	10.485	2.127
17	53.748	3.677
18	57.052	3.872
19	20.123	1.713
20	33.229	3.661

The test for significant difference between the standard deviation measures and the geometric measure of variation showed that there was a significant difference between the geometric measures estimates and the standard deviation estimates (t=2.9501, df=19, p-value=0.004). The test showed that the average estimates for the geometric measure of variation from the mean were significantly smaller than those of standard deviation (p-value=0.008).

The test results for the efficiency of the geometric measure in comparison to standard deviation, based on the four measures of efficiency were as illustrated in Table 4.38;

Table 4.38: Efficiency Measures for Large Peaked Sample

Measure of Efficiency	Standard Deviation	Geometric Measure of Variation
Bias	25.166	1.478
Mean Squared Error	2944.155	10.023
Relative Efficiency	293.740	

The results in Table 4.38 illustrated that geometric measure had a smaller bias compared to standard deviation. It also had a smaller Mean Squared Error compared to standard deviation. Comparing the MSE of the two measures through relative efficiency, it is observed that the relative efficiency was more than 1, meaning that geometric measure was more efficient than the standard deviation in estimating the average variation about the mean for large peaked samples.

CHAPTER FIVE

SUMMARY, CONCLUSION, AND RECOMMENDATION

5.1 Summary

Based on the derivation of the geometric measure of variation from the mean, the application of the function on simulated datasets and functions, and the tests on the efficiency of the geometric measure of variation from the mean in comparison to standard deviation, the following is a summary of the findings made in chapter four in line with the objectives;

5.1.1 Formulation of geometric measure of variation

In the derivation of the geometric measures of variation from the mean for un-weighted datasets, for discrete and continuous un-weighted datasets the geometric measure of variation from the mean was derived as;

$$G_u = \begin{cases} \exp(\frac{1}{n}\sum_{i=1}^{P} \ln(|d_i|)) & d_1 \neq d_2 \neq d_3 \neq \ldots \neq d_n \neq 0 \\ 0 & d_1 = d_2 = d_3 = \ldots = d_n = 0 \end{cases}$$

For both discrete and continuous weighted datasets, the geometric measure of variation from the mean was derived as;

$$G_w = \begin{cases} \exp\left(\frac{1}{\sum_{i=1}^{n} \varsigma_i} \sum_{i=1}^{p} \varsigma_i \ln(|d_i|) \right) & d_1 \neq d_2 \neq d_3 \neq \ldots \neq d_n \neq 0 \\ 0 & d_1 = d_2 = d_3 = \ldots = d_n = 0 \end{cases}$$

For probability mass functions, the geometric measure of variation from the mean was derived as;

$$G_{pm}=\begin{cases}\exp\left(\sum_{i=1}^{n}\varsigma(v_i)\ln|d_i|\right) & d_1\neq d_2\neq d_3\neq\ldots\neq d_n\neq 0\\ 0 & d_1=d_2=d_3=\ldots=d_n=0\end{cases}$$

While for probability density functions, the geometric measure of variation from the mean was derived as;

$$G_{pd}=\begin{cases}\exp\left(\int_{a}^{b}\varsigma(v)\ln|v-\bar{v}|.dv\right) & v\neq\bar{v}\\ 0 & v=\bar{v}\end{cases}$$

5.1.2 Application of function on simulated datasets and empirical data

For un-weighted datasets, six sets of data were simulated each of size 10 from three discrete distributions (Bernoulli, Binomial and Geometric) and three continuous distributions (Normal, Chi-square, and F-distribution). The function for the geometric measure of variation from the mean for un-weighted datasets was used to estimate the average variation from the arithmetic mean for the six datasets. It was determined that the geometric measure of variation from the mean function could be used in estimating the average variation about the mean for both discrete and continuous un-weighted datasets.

For weighted datasets, six sets of data were simulated each of size 100 from three discrete distributions (Bernoulli, Binomial and Geometric) and three continuous distributions (Normal, Chi-square, and F-distribution), which were then grouped to form weighted datasets with frequencies as the weights. The geometric measure of variation from the mean for weighted datasets was then used to estimate the average variation from the arithmetic mean for the six datasets. It was determined that the geometric measure of variation from the mean function could be used in estimating the average variation about the mean for both discrete and continuous weighted datasets.

Hypothetical probability mass functions with empirical and theoretical distributions were generated (tossing coin gambling game, dice, Bernoulli, binomial, geometric and Poisson distribution). The function for the geometric measure of variation from the mean for probability mass functions was then used to estimate the average deviation from the mean for the empirical and theoretical distributions. It was determined that the functions could be used in estimating the average deviation from the mean for probability mass functions.

Hypothetical probability density functions with empirical and theoretical distributions were generated (simple probability density function, uniform distribution, exponential distribution, and standard normal distribution). The function for the geometric measure of variation from the mean for probability density functions was then used to estimate the average deviation from the mean for the empirical and theoretical distributions. It was determined that the functions could only be used in estimating the average deviation from the mean for probability density functions with finite intervals. However, for those with infinite intervals, it was not possible to estimate a finite average deviation from the mean. It was also determined that it is sometimes complicated to estimate the average deviation from the mean for geometric measure because of the complication involved during the partial integration which sometimes is indefinite.

5.1.3 Test for efficiency in comparison to standard deviation

The sample estimates for the geometric measure of variation for small samples of size 10 and large samples of size 200, were tested against standard deviation estimates to test if the estimates given by the geometric measure of variation from the mean were significantly smaller than those given by standard deviation. A paired-samples t-test was used to test the difference of the averages of the estimates given by the two measures. It was determined that estimates given by the geometric measure of variation from the mean were significantly smaller than the estimates given by standard deviation.

Apart from testing a difference in the estimates of geometric measure of variation and those of standard deviation, a test for the efficiency of the geometric measure in comparison to standard deviation in estimating the average deviation from the mean for discrete and continuous random variables was carried out. The efficiency was tested based on three measures of efficiency (Bias, Mean Squared Error and relative efficiency). The test was carried out on small and large twenty samples of sizes 10 and 200 respectively, from both discrete and continuous distributions. For the discrete distributions, the study simulated data for three types of discrete distributions (Binary discrete, countable discrete and geometric discrete). Similarly, for continuous distributions, the study simulated data for three continuous distributions (normal, skewed and peaked distributions). The summary of the analysis is as illustrated in Table 5.1 and 5.2. The ticked box marks the most efficient measure of variation based on the specified measure of variation. If both boxes are marked then the two measures are equally efficient.

5.1: Small Samples Efficiency Test

Small Samples	**Measure**	**RE**	**BIAS**	**MSE**	**Most Efficient**
Binary Discrete	Geometric Measure				
	Standard Deviation	✖	✖	✖	✖
Geometric Discrete	Geometric Measure				
	Standard Deviation	✖	✖	✖	✖
Countable Discrete	Geometric Measure	✖		✖	✖
	Standard Deviation		✖		

Normal Continuous	Geometric Measure	✖	✖	✖	✖
	Standard Deviation				
Skewed Continuous	Geometric Measure	✖	✖	✖	✖
	Standard Deviation				
Peaked Continuous	Geometric Measure	✖	✖	✖	✖
	Standard Deviation				

Table 5.2: Large Samples Efficiency Test

Large Samples	**Measure**	**RE**	**BIAS**	**MSE**	**Most Efficient**
Binary Discrete	Geometric Measure				
	Standard Deviation	✖	✖	✖	✖
Geometric Discrete	Geometric Measure	✖		✖	✖
	Standard Deviation		✖		
Countable Discrete	Geometric Measure				
	Standard Deviation	✖	✖	✖	✖
Normal Continuous	Geometric Measure	✖		✖	✖
	Standard Deviation		✖		
Skewed Continuous	Geometric Measure	✖	✖	✖	✖
	Standard Deviation				
Peaked Continuous	Geometric Measure	✖	✖	✖	✖
	Standard Deviation				

5.2 Conclusion

Based on the results obtained in chapter four and summarized in section 5.1, it can be concluded that the geometric measure of variation from the mean can be used to estimate the average deviation from the mean for un-weighted datasets, weighted datasets, probability mass and probability density functions with finite intervals. The geometric measure of variation from the mean was determined to give smaller estimates for average variation about the mean compared to standard deviation. In terms of efficiency, the geometric measure of variation about the mean was determined to be more efficient in estimating the average variation about the mean for geometric, skewed and peaked datasets. This is because geometric and peaked datasets are very much prone to outliers, a factor that interferes with the standard deviation estimation. It is also efficient than standard deviation in estimating the average deviation from the mean for skewed datasets, this is because standard deviation assumes normality of data during estimation. In conclusion, the study was able to formulate a new measure of variation about the mean that overcame the weaknesses of the existing measures of variation about the mean.

5.3 Recommendation

Based on the findings made in the study, it is recommended that geometric measure of variation from the mean to be used in various research in estimating the average deviation from the mean for geometric, skewed, peaked and datasets with outliers because the function is more efficient in estimating the average deviation about the mean than standard deviation. This is because the geometric averaging technique introduced by the measure is not affected by outliers unlike the arithmetic averaging techniques used by standard deviation which is affected by outliers. The study also recommends that further research be carried out on the application of the geometric measure of variation on probability density functions especially in the estimation of the average deviation from the mean for probability density functions with infinite intervals.

5.4 Area for Further Research

Further research and analysis can be carried out in the following areas;

i. Comparative analysis of the geometric measure of variation about the mean against other measures about the mean such as mean deviation and variance.

ii. Application of the geometric measure of variation about the mean on discrete datasets using other measures of central tendency such as median and mode.

iii. Further analysis of the geometric measure of variation about the mean such as the construction of confidence intervals for measures of central tendency.

iv. Further research need to be carried out on the use of geometric measure to estimate average variation for probability density functions, due to the complexity involved in the integration of such functions to obtain the estimates.

REFERENCES

Ahn, S., & Fessler, J. A., (2003). *Standard Errors of Mean, Variance, and Standard Deviation Estimators*. EECS Department. The University of Michigan. U.S.A

Altman, D. G., & Bland, J. M. (2005). Standard deviations and standard errors. *BMJ* 331.

Bhardwaj, A., (2013). Comparative Study of Various Measures of Dispersion. *Journal of Advances in Mathematics*. 1(1).

Buckland, S. T., A. C. Studeny, A. E. Magurran, J. B. Illian, and S. E. Newson. (2011). *The geometric mean of relative abundance indices: a biodiversity measure with a difference*. Ecosphere 2(9):100. Available at http://dx.doi:10.1890/ES11-00186.1

Clark, P. L., (2012). Number Theory: A Contemporary Introduction. Available at http://math.uga.edu/~pete/4400FULL.pdf

Deshpande, S., Gogtay, N. J., Thatte, U. M., (2016). Measures of Central Tendency and Dispersion. *Journal of the Association of Physicians of India*.64. .

Grechuk, B., Molyboha, A., & Zabarankin M., (2011). Mean-Deviation Analysis in The Theory of Choice. *Risk Analysis: An International Journal*. 32(8).

Hu, S., (2010). *Simple Mean, Weighted Mean, or Geometric Mean*?. Presented at the 2010 ISPA/SCEA Joint Annual Conference and Training Workshop. Tecolote Research INC. Available at http://www.iceaaonline.com/ready/wp-content/uploads/2017/09/MET08A-Hu.pdf

Kum, S., & Lim, Y., (2012). *A Geometric Mean of Parameterized Arithmetic and Harmonic Means of Convex Functions*. Hindawi Publishing Corporation. 20

Lawson, J. D., & Lim, Y., (2001). The Geometric Mean, Matrices, Metrics, and More. *The American Mathematical Monthly*.

Lee, D., In, J., & Lee, S.,(2015). Standard deviation and standard error of the mean. *Korean Journal of anesthesiology*. 68. 220-3. 10.4097/kjae.2015.68.3.220.

Leys, C., Klein, O., Bernard, P., & Licata, L., (2013). Detecting outliers: Do not use standard deviation around the mean, use absolute deviation around the median. *Journal of Experimental Social Psychology* 49 (pg.764–766).

Manikandan, S., (2016). Measures of dispersion. *Journal of Pharmacology and Pharmacotherapeutics*. 2(4).

McAlister, D., (1879). *The Law of Geometric Mean*. Royal Society. 29 (pg. 367-376). Available at https://www.jstor.org/stable/113784

Mindlin, D., (2011). On the Relationship between Arithmetic and Geometric Returns. *Cdi Advisors Research. LLC*. Available at http://dx.doi.org/10.2139/ssrn.2083915

Mohini, P. B., & Prajakt, J. B., (2012). What to use to express the variability of data: Standard deviation or standard error of mean?. *Perspect Clin Res* 3(3): 113–116.

Raymond, J., (2015). *Measures of Variation from Statistical Analysis in the Behavioral Sciences*. Kendall Hunt Publishing.

Roberson, Q. M., Sturman, M. C., & Simons, T. L., (2007). *Does the Measure of Dispersion Matter in Multilevel Research? A Comparison of the Relative Performance of Dispersion Indexes*. Cornell University School of Hotel Administration. The Scholarly Commons.

Roenfeldt, K., (2018). *Better than average: Calculating Geometric Means Using SAS*. Henry. M. Foundation for the Advancement of Military Medicine.

Schuetter, J. (2007). Chapter 1. In J. Schuetter, *measures of dispersion* (pp. 45-54).

Thenwall, M. (2018). *The precision of the arithmetic mean, geometric mean and percentiles for citation data: An experimental simulation modeling approach*. Statistical Cybermetrics Research Group, School of Mathematics and Computer Science, University of Wolverhampton, Wulfruna Street, Wolverhampton, UK.