INFINITUDE OF PRIME p such that $ap + b$ is Prime with $\gcd(a, b) = 1$

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Abstract: In 1904, Dickson [6] stated a very important conjecture. Now people call it Dickson's conjecture. In 1958, Schinzel and Sierpinski [3] generalized Dickson's conjecture to the higher order integral polynomial case. However, they did not generalize Dickson's conjecture to the multivariable case. In 2006, Green and Tao [9] considered Dickson's conjecture in the multivariable case and gave directly a generalized Hardy-Littlewood estimation. But, the precise Dickson's conjecture in the multivariable case does not seem to have been formulated. In this paper, based on the idea in [8] we introduce an interesting class of prime numbers to solve the dickson conjecture Although this article does not solve the dickson conjecture but it solves a problem that is similar to the Dickson conjecture. the problem is stated as follows being given two coprime integers a, b there is an infinity of prime numbers p such that ap+b is prime. This type of prime numbers we call it Bado-Tiemoko prime numbers . We intend to generalize this result but for the moment we speculate that given a family $(a_i, b_i)_{1\leq i\leq k}$ such that $\gcd(a_i, b_i) = 1, \forall 1 \leq i \leq k$ there is an infinity of prime numbers p such that $a_i p + b_i$ is prime for $\forall 1 \leq i \leq k$ Let q_a be the smallest prime number dividing a and $\omega(q_a)$ its order by arranging the prime numbers in ascending order.Let $\beta(n)$ the number of

Bado-Tiemoko prime less than n and

$$\eta_{s,i} = \frac{(p_i - 1)^{\delta_{p_i}(a)} \prod_{k=1}^{s} (p_{i_k} - 1)^{\delta_{p_{i_k}}(a)}}{\phi(a) \prod_{k=1}^{s} (p_{i_k} - 1)(p_i - 1) \prod_{k=1}^{s} p_{i_k}^{\delta_{p_{i_k}}(a)} p_i^{\delta_{p_i}(a)}}$$

$$\mu(r) = \frac{1}{\phi(a)} [\sum_{s=1}^{\omega(q_a)} (-1)^{s-1} \sum_{1 \leq i_1 < i_2 ...... < i_s \leq \omega(q_a)} \prod_{i=\omega(q_a)+1}^{r} (1 - \eta_{s,i})]$$

We show that :

$$\frac{\beta(n) - \beta(\sqrt{n})}{\Pi(an+b)} = -\frac{2}{q_a \phi(a)} + \frac{1}{\phi(a)} + \frac{4e^{-\gamma} c_2(n) \lambda_n(a) \epsilon(a)}{\phi(a) \ln n} + \bigcirc(\frac{1}{\ln^2 n}) - \mu(r), \forall a \not\equiv 0[2]$$

$$\frac{\beta(n) - \beta(\sqrt{n})}{\Pi(an+b)} = \frac{4e^{-\gamma} c_2(n) \lambda_n(a) \epsilon(a)}{\phi(a) \ln n} + \bigcirc(\frac{1}{\ln^2 n}) - \mu(r), \forall a \equiv 0[2]$$

Keywords: Dickson conjecture, Chebotarev theorem,Bado-Tiemoko Prime theorem, Mertens formula

1 INTRODUCTION

One of the difficult problems of number theory is to study the infinity of prime numbers by an affine transformation. It is in this sense several speculations such as Dickson's have been formulated. Throughout our study we will designate by a special affine function the affine function with coprime coefficient for example function f is called special affine if it is of the form $f(m) = am + b$ with a, b integers such that $\gcd(a, b) = 1$ In consideration of this form of function we introduce a class of prime number that we will designate by Bado-Tiemoko prime's numbers whose image by an affine function is prime . We have formulated a very interesting conjecture on this class of prime numbers that we have solved... The resolution generalizes several conjectures that we have solved... For example that of sophie Germain, twin prime numbers and even gap prime numbers... We are thinking in our future research to generalize the method in dimension k in order to justify the veracity of Dickson's conjecture. The tools we used are the same tools we are used to using.

1.1 Notations

Denote by $\mathcal{P}_n$ the set of prime numbers less that n $v_p(n) = s$ if $p^s \mid n$ and $p^{s+1} \nmid n$

$$\Omega(n) = \sum_{p\in\mathcal{P}_n, p|n} v_p(n)$$

$\delta_p(n) = 1$ if p is a prime divisor of n and 0 otherwise . $\delta(n) = 1$ if n is prime and 0 otherwise.

$$C_n = \{m \leq n, \Omega(m) \geq 2\}$$

$$\overbrace{\mathcal{P}_{an+b}} = \{p \in \mathcal{P}_{an+b} : p \equiv b[a]\}$$

$$A_n = \{am + b \in \overbrace{\mathcal{P}_{an+b}} : m \in C_n\}$$

$$T_n = \{2p, 3p, 4p, ...[\frac{n}{p}]p\}$$

$$\alpha(n) = card(A_n)$$

$$\beta(n) = card(\overbrace{\mathcal{P}_{an+b}} \backslash A_n)$$

$\omega(p) = i$ if p is ith prime number.

1.2 Principle of Proof

$$C_n = \bigcup_{p\in\mathcal{P}_{\sqrt{n}}} T_n$$

$$\alpha(n) + \beta(n) = \Pi(an + b, a, b)$$

f_n be the function defined as : $\begin{array}{cccc} f_n & : & C_n & \to & \mathbb{N} \\ & & m & \mapsto & am + b \end{array}$

$$f_n(C_n) = \bigcup_{p\in\mathcal{P}_{\sqrt{n}}} f_n(T_n)$$

Where

$$f_n(T_n) = \{2pa + b, 3pa + b, 4pa + b, ...a[\frac{n}{p}]p + b\}$$

The elements of $f_n(A_{2p})$ are in an arithmetic progression of reason ap. We will evaluate the quantity of prime numbers in $f_n(C_n)$ by applying the principle -exclusion of Moivre and Chebotarev -Artin theorem in each set $f_n(T_{2p})$ We adapt the method in [8] and we obtain

$$\forall k > 2, \bigcap_{j=1, p_{i_j}\in\mathcal{P}_{\sqrt{n}}}^{k} f_n(T_{2p_{i_j}}) = \{am\prod_{j=1}^{k} p_{i_j} + b, 1 \leq m \leq [\frac{n}{\prod_{j=1}^{k} p_{i_j}}]\}$$

The elements of this set are in arithmetic progression of reason $a\prod_{j=1}^{k} p_{i_j}$ The hypothesis of application of Chebotarev-Artin's theorem is justified .

2 Theorem

Let q_a be the smallest prime number dividing a and $\omega(q_a)$ its order by arranging the prime numbers in ascending order.Let $\beta(n)$ the number of Bado's prime less than n and

$$\eta_{s,i} = \frac{(p_i - 1)^{\delta_{p_i}(a)} \prod_{k=1}^{s} (p_{i_k} - 1)^{\delta_{p_{i_k}}(a)}}{\phi(a) \prod_{k=1}^{s} (p_{i_k} - 1)(p_i - 1) \prod_{k=1}^{s} p_{i_k}^{\delta_{p_{i_k}}(a)} p_i^{\delta_{p_i}(a)}}$$

$$\mu(r) = \frac{1}{\phi(a)} [\sum_{s=1}^{\omega(q_a)} (-1)^{s-1} \sum_{1 \leq i_1 < i_2 ...... < i_s \leq \omega(q_a)} \prod_{i=\omega(q_a)+1}^{r} (1 - \eta_{s,i})]$$

Then

$$\frac{\beta(n) - \beta(\sqrt{n})}{\Pi(an+b)} = -\frac{2}{q_a \phi(a)} + \frac{1}{\phi(a)} + \frac{4e^{-\gamma} c_2(n) \lambda_n(a) \epsilon(a)}{\phi(a) \ln n} + \bigcirc(\frac{1}{\ln^2 n}) - \mu(r), \forall a \not\equiv 0[2]$$

$$\frac{\beta(n) - \beta(\sqrt{n})}{\Pi(an+b)} = \frac{4e^{-\gamma} c_2(n) \lambda_n(a) \epsilon(a)}{\phi(a) \ln n} + \bigcirc(\frac{1}{\ln^2 n}) - \mu(r), \forall a \equiv 0[2]$$

2.1 Proof

Let us denote as the function ϱ which represents the proportion of prime numbers which appear in a given set over prime numbers less than n and $q_a = \min(p \in \mathcal{P}_{\sqrt{n}} : p \mid a)$ Since

$$\bigcup_{p \in \mathcal{P}_{\sqrt{n}}} f_n(T_{2p}) = \bigcup_{p \leq q_a} f_n(T_{2p}) \cup \bigcup_{q_a < p \leq \sqrt{n}} f_n(T_{2p})$$

With regard to the principle of inclusion -exclusion of Moivre we can write:

$$\varrho(\bigcup_{p \in \mathcal{P}_{\sqrt{n}}} f_n(T_{2p})) = \sum_{s=1}^{\omega(q_a)} (-1)^{s-1} \sum_{1 \leq i_1 < i_2 ...... < i_s \leq \omega(q_a)} \varrho(\bigcap_{j=1, p_{i_j} \in \mathcal{P}_{\sqrt{n}}}^{s} f_n(T_{2p_{i_j}})) + R(r) - \varrho_2(n, a)$$

with

$$R(r) = \sum_{s=\omega(q_a)+1}^{r} (-1)^{s-1} \sum_{\omega(q_a)+1 \leq i_1 < i_2 ...... < i_s \leq r} \varrho(\bigcap_{j=\omega(q_a)+1, p_{i_j} \in \mathcal{P}_{\sqrt{n}}}^{s} f_n(T_{2p_{i_j}}))$$

Moreover, we have

$$\varrho(f_n(C_n)) = \frac{card(A_n)}{\Pi(an+b)} = \frac{\alpha(n)}{\Pi(an+b)}$$

$$\varrho_2(n,a) = \varrho(\bigcup_{p\leq q_a} f_n(T_{2p}) \cap \bigcup_{q_a<p\leq\sqrt{n}} f_n(T_{2p})) = \varrho(f_n(\bigcup_{p\leq q_a} T_{2p} \cap \bigcup_{q_a<p\leq\sqrt{n}} T_{2p}))$$

Since

$$\bigcup_{p\leq q_a} T_{2p} \cap \bigcup_{q_a<p\leq\sqrt{n}} T_{2p} = \bigcup_{p\leq q_a} \{2p, 3p, ..., [\frac{n}{p}]p\} \cap \bigcup_{q_a<p\leq\sqrt{n}} \{2p, 3p, .., [\frac{n}{p}]p\}$$

$$\bigcup_{p\leq q_a} T_{2p} \cap \bigcup_{q_a<p\leq\sqrt{n}} T_{2p} = \bigcup_{i=1}^{\omega(q_a)} \bigcup_{j=\omega(q_a)+1}^{r} \{mp_ip_j, 1 \leq m \leq [\frac{n}{p_ip_j}]\}$$

$$F_s = \bigcup_{j=\omega(q_a)+1}^{r} \{m \prod_{k=1}^{s} p_{i_k}p_j, 1 \leq m \leq [\frac{n}{\prod_{k=1}^{s} p_{i_k}p_j}]\}$$

$$\varrho_2(n,a) = \sum_{s=1}^{\omega(q_a)} (-1)^{s-1} \sum_{1\leq i_1<i_2......<i_s\leq\omega(q_a)} \varrho(f_n(F_s))$$

$$\varrho(f_n(F_s)) = \sum_{l=\omega(q_a)+1}^{r} (-1)^{l-1} \sum_{\omega(q_a)+1\leq i_1<i_2......<i_l\leq r} \varrho(f_n(G_l))$$

$$G_l = \{m \prod_{k=1}^{s} p_{i_k} \prod_{q=\omega(q_a)+1}^{l} p_{j_q}, 1 \leq m \leq [\frac{n}{\prod_{k=1}^{s} p_{i_k} \prod_{q=\omega(q_a)+1}^{l} p_{j_q}}]\}$$

$$f_n(G_l) = \{am \prod_{k=1}^{s} p_{i_k} \prod_{q=\omega(q_a)+1}^{l} p_{j_q} + b, 1 \leq m \leq [\frac{n}{\prod_{k=1}^{s} p_{i_k} \prod_{q=\omega(q_a)+1}^{l} p_{j_q}}]\}$$

According to Chebotarev's theorem -Artin more precisely the corollary we have

$$\varrho(\bigcap_{j=1,p_{i_j}\in\mathcal{P}_{\sqrt{n}}}^{s} f_n(T_{2p_{i_j}})) = \frac{1}{\phi(a\prod_{j=1}^{s} p_{i_j}), p_{i_j} \in \mathcal{P}_{\sqrt{n}}} + h(an+b), \forall k > 2$$

$$\varrho(f_n(G_l)) = \frac{1}{\phi(a\prod_{k=1}^{s} p_{i_k} \prod_{q=\omega(q_a)+1}^{l} p_{j_q})} + h(an+b)$$

$$\varrho(f_n(T_{2p_i}), p_i \in \mathcal{P}_{\sqrt{n}}) = \frac{1}{\phi(ap_i)} - \frac{\delta(ap_i+b)}{\Pi(an+b)} + h(an+b), \forall i \geq 1$$

where h represents the error of the proportion estimation. Also we have

$$\sum_{k=1,p_k\mathcal{P}_{\sqrt{n}}}^{r} \delta(ap_k+b) = \sum_{p\in\mathcal{P}_{p_r},\Omega(ap+b)=1} 1 = \beta(p_r) = \beta(\sqrt{n})$$

and applying the useful lemma in [8], we have:

$$\frac{\alpha(n)}{\Pi(an+b)} = h(an+b) - \frac{\beta(\sqrt{n})}{\Pi(an+b)} + \sum_{s=1}^{\omega(q_a)} (-1)^{s-1} \sum_{1\le i_1<i_2......<i_s\le\omega(q_a)} \frac{1}{\phi(a\prod_{j=1}^{s} p_{i_j})} + R(r) - \varrho_2(n,a)$$

with

$$R(r) = \sum_{s=\omega(q_a)+1}^{r} (-1)^{s+\omega(q_a)-1} \sum_{\omega(q_a)+1\le i_1<i_2......<i_s\le r} \frac{1}{\phi(a\prod_{j=\omega(q_a)+1}^{s} p_{i_j})}$$

Since

$$\phi(a\prod_{j=1}^{s} p_{i_j}) = \phi(a)\phi(\prod_{j=1}^{s} p_{i_j})\frac{\gcd(a,\prod_{j=1}^{s} p_{i_j})}{\phi(\gcd(a,\prod_{j=1}^{s} p_{i_j}))}$$

Then

$$\phi(a\prod_{j=1}^{s} p_{i_j}) = \phi(a)\prod_{j=1}^{s}(p_{i_j}-1)\frac{\prod_{j=1}^{s} p_{i_j}^{\delta_{p_{i_j}}(a)}}{\prod_{j=1}^{s}(p_{i_j}-1)^{\delta_{p_{i_j}}(a)}}$$

$$\phi(a\prod_{k=1}^{s} p_{i_k}\prod_{q=\omega(q_a)+1}^{l} p_{j_q}) = \phi(a)\phi(\prod_{k=1}^{s} p_{i_k})\phi(\prod_{q=\omega(q_a)+1}^{l} p_{j_q})\frac{\gcd(a,\prod_{k=1}^{s} p_{i_k}\prod_{q=\omega(q_a)+1}^{l} p_{j_q})}{\phi(\gcd(a,\prod_{k=1}^{s} p_{i_k}\prod_{q=\omega(q_a)+1}^{l} p_{j_q}))}$$

Let

$$b_{l,s} = \phi(a\prod_{k=1}^{s} p_{i_k}\prod_{q=\omega(q_a)+1}^{l} p_{j_q})$$

$$b_{l,s} = \phi(a)\prod_{k=1}^{s}(p_{i_k}-1)\prod_{q=\omega(q_a)+1}^{l}(p_{j_q}-1)\frac{\prod_{k=1}^{s} p_{i_k}^{\delta_{p_{i_k}}(a)}\prod_{q=\omega(q_a)+1}^{l} p_{j_q}^{\delta_{p_{j_q}}(a)}}{\prod_{k=1}^{s}(p_{i_k}-1)^{\delta_{p_{i_k}}(a)}\prod_{q=\omega(q_a)+1}^{l}(p_{j_q}-1)^{\delta_{p_{j_q}}(a)}}$$

and

$$a_{i_j} = \frac{(p_{i_j-1})^{\delta_{p_{i_j}}(a)}}{(P_{i_j}-1)p_{i_j}^{\delta_{p_{i_j}}(a)}}$$

$$\frac{\alpha(n)}{\Pi(an+b)} = h(an+b) - \frac{\beta(\sqrt{n})}{\Pi(an+b)} + G(\omega(q_a)) + R(r) - \varrho_2(n,a)$$

Where

$$G(\omega(q_a)) = \frac{1}{\phi(a)}\sum_{s=1}^{\omega(q_a)}(-1)^{s-1}\sum_{1\le i_1<i_2......<i_s\le\omega(q_a)}\prod_{j=1}^{s} a_{i_j}$$

and

$$R(r)=\frac{1}{\phi(a)}\sum_{s=\omega(q_a)+1}^{r}(-1)^{s+\omega(q_a)-1}\sum_{\omega(q_a)+1\le i_1<i_2......<i_s\le r}\prod_{j=\omega(q_a)+1}^{s}a_{i_j}$$

$$G(\omega(q_a))=\frac{1}{\phi(a)}\sum_{s=1}^{\omega(q_a)-1}(-1)^{s-1}\sum_{1\le i_1<i_2......<i_s\le\omega(q_a)-1}\prod_{j=1}^{s}a_{i_j}+\frac{1}{\phi(a)}I(\omega(q_a))$$

$$I(\omega(q_a))=a_{\omega(q_a)}\sum_{s=1}^{\omega(q_a)-1}(-1)^{s-1}\sum_{1\le i_1<i_2......<i_s\le\omega(q_a)-1}\prod_{j=1}^{s}a_{i_j}+a_{\omega(q_a)}$$

By the useful lemma in [8]

$$I(\omega(q_a))=2a_{\omega(q_a)}-a_{\omega(q_a)}\prod_{i=1}^{\omega(q_a)-1}(1-a_i)$$

Then

$$G(\omega(q_a))=\frac{1}{\phi(a)}[[1-\prod_{i=1}^{\omega(q_a)-1}(1-a_i)]+2a_{\omega(q_a)}-a_{\omega(q_a)}\prod_{i=1}^{\omega(q_a)-1}(1-a_i)]$$

and

$$R(r)=\frac{1}{\phi(a)}[1-\prod_{i=\omega(q_a)+1}^{r}(1-a_i)]$$

$$1-a_i=\frac{p_i-1}{p_i},\forall p_i\mid a,\forall i\neq\omega(q_a)$$

$$1-a_i=\frac{p_i-2}{p_i-1},\forall p_i\nmid a$$

$$a_{\omega(q_a)}=\frac{1}{q_a}$$

Hence

$$\frac{\alpha(n)}{\Pi(an+b)}=h(an+b)-\frac{\beta(\sqrt{n})}{\Pi(an+b)}+\frac{1}{\phi(a)}-[\frac{1}{\phi(a)}+\frac{a_{\omega(q_a)}}{\phi(a)}]\prod_{i=1}^{\omega(q_a)-1}(1-a_i)+\frac{2a_{\omega(q_a)}}{\phi(a)}+R(r)-\varrho_2(n,a)$$

$$\prod_{i=\omega(q_a)+1}^{r}(1-a_i)=\prod_{q_a+1\le p\le\sqrt{n},p\mid a}\frac{p-1}{p}\prod_{q_a+1\le p\le\sqrt{n},p\nmid a}\frac{p-2}{p-1}$$

$$\prod_{i=\omega(q_a)+1}^{r}(1-a_i) = \prod_{q_a+1\le p\le\sqrt{n},p|a}\frac{p-1}{p}\prod_{q_a+1\le p\le\sqrt{n},p|a}\frac{p-1}{p-2}\prod_{q_a+1\le p\le\sqrt{n}}\frac{p-2}{p-1}$$

Let

$$c_2(n) = \prod_{p=3}^{\sqrt{n}}\frac{p(p-2)}{(p-1)^2} = \prod_{p=3}^{\sqrt{n}}\frac{p}{p-1}\prod_{p=3}^{\sqrt{n}}\frac{p-2}{p-1}$$

Then

$$\prod_{p=q_a+1}^{\sqrt{n}}\frac{p-2}{p-1} = 2\prod_{p=2}^{\sqrt{n}}[1-\frac{1}{p}]c_2(n)\epsilon(a)$$

where $\epsilon(a) = \prod_{p=3}^{q_a}\frac{p-1}{p-2}$ if $q_a \ge 3$ and 1 if not

We put

$$\lambda_n(a) = \prod_{q_a+1\le p\le\sqrt{n},p|a}\frac{(p-1)^2}{p(p-2)}$$

By Mertens third formula we have

$$\prod_{p=2}^{\sqrt{n}}(1-\frac{1}{p}) = \frac{e^{-\gamma}}{\ln\sqrt{n}}(1+\bigcirc(\frac{1}{\ln\sqrt{n}})) = \frac{2e^{-\gamma}}{\ln n}(1+\bigcirc(\frac{1}{\ln n}))$$

Hence

$$\prod_{i=\omega(q_a)+1}^{r}(1-a_i) = \frac{4e^{-\gamma}c_2(n)\lambda_n(a)\epsilon(a)}{\ln n}[1+\bigcirc(\frac{1}{\ln n})]$$

By the definition of q_a we have

$$\prod_{i=1}^{\omega(q_a)-1}(1-a_i) = \prod_{p<q_a}\frac{p-2}{p-1}$$

So

$$\frac{\alpha(n)}{\Pi(an+b)} = h(an+b) - \frac{\beta(\sqrt{n})}{\Pi(an+b)} + \frac{1}{\phi(a)} - [\frac{1}{\phi(a)} + \frac{a_{\omega(q_a)}}{\phi(a)}]\prod_{p<q_a}\frac{p-2}{p-1} + \frac{2a_{\omega(q_a)}}{\phi(a)} + R(r) - \varrho_2(n,a)$$

The expression of

$$R(r) = \frac{1}{\phi(a)}[1-\frac{4e^{-\gamma}c_2(n)\lambda_n(a)\epsilon(a)}{\ln n}[1+\bigcirc(\frac{1}{\ln n})]]$$

Since

$$\frac{\alpha(n)}{\Pi(an+b)} = \frac{\Pi(an+b,a,b)}{\Pi(an+b)} - \frac{\beta(n)}{\Pi(an+b)}$$

By Chebotarev -Artin theorem we have :

$$\frac{\Pi(an+b,a,b)}{\Pi(an+b)} = \frac{1}{\phi(a)} + h(an+b)$$

Then

$$\frac{\beta(n)-\beta(\sqrt{n})}{\Pi(an+b)} = [\frac{1}{\phi(a)} + \frac{a_{\omega(q_a)}}{\phi(a)}]\prod_{p<q_a}\frac{p-2}{p-1} - \frac{2a_{\omega(q_a)}}{\phi(a)} - R(r) + \varrho_2(n,a)$$

$$\frac{\beta(n)-\beta(\sqrt{n})}{\Pi(an+b)} = D(a,n) + \bigcirc(\frac{1}{\ln^2 n}) + \varrho_2(n,a)$$

With

$$D(a,n) = [\frac{1}{\phi(a)} + \frac{a_{\omega(q_a)}}{\phi(a)}]\prod_{p<q_a}\frac{p-2}{p-1} - \frac{2a_{\omega(q_a)}}{\phi(a)} - \frac{1}{\phi(a)} + \frac{4e^{-\gamma}c_2(n)\lambda_n(a)\epsilon(a)}{\phi(a)\ln n} + \varrho_2(n,a)$$

In obvious manner

$$\prod_{p<q_a}\frac{p-2}{p-1} = 0, \forall q_a \geq 3$$

Finally

$$\frac{\beta(n)-\beta(\sqrt{n})}{\Pi(an+b)} = -\frac{2a_{\omega(q_a)}}{\phi(a)} - \frac{1}{\phi(a)} + \frac{4e^{-\gamma}c_2(n)\lambda_n(a)\epsilon(a)}{\phi(a)\ln n} + \bigcirc(\frac{1}{\ln^2 n}) + \varrho_2(n,a), \forall q_a \geq 3$$

$$\frac{\beta(n)-\beta(\sqrt{n})}{\Pi(an+b)} = -\frac{a_{\omega(q_a)}}{\phi(a)} + \frac{4e^{-\gamma}c_2(n)\lambda_n(a)\epsilon(a)}{\phi(a)\ln n} + \bigcirc(\frac{1}{\ln^2 n}) + \varrho_2(n,a)$$

2.2 EVALUATION OF $\varrho_2(n,a)$

$$\varrho(f_n(F_s)) = \sum_{l=\omega(q_a)+1}^{r}(-1)^{l+\omega(q_a)-1}\sum_{\omega(q_a)+1\leq i_1<i_2......<i_l\leq r}\prod_{q=\omega(q_a)+1}^{l}\eta_{s,j_q}$$

With

$$b_{l,s} = \prod_{q=\omega(q_a)+1}^{l}\frac{1}{\eta_{s,j_q}}$$

$$\varrho(f_n(F_s)) = \frac{1}{\phi(a)}[1 - \prod_{i=\omega(q_a)+1}^{r}(1-\eta_{s,i})]$$

then

$$\varrho_2(n,a) = \sum_{s=1}^{\omega(q_a)} (-1)^{s-1} \sum_{1\leq i_1<i_2......<i_s\leq \omega(q_a)} \frac{1}{\phi(a)}[1 - \prod_{i=\omega(q_a)+1}^{r} (1-\eta_{s,i})]$$

Since $\forall q_a \geq 3$

$$\sum_{s=1}^{\omega(q_a)} (-1)^{s-1} \sum_{1\leq i_1<i_2......<i_s\leq \omega(q_a)} \frac{1}{\phi(a)} = -\frac{1}{\phi(a)} \sum_{s=1}^{\omega(q_a)} (-1)^s C^s_{\omega(q_a)} = -\frac{1}{\phi(a)}[(1-1)^{\omega(q_a)}-1] = \frac{1}{\phi(a)}$$

and if $q_a = 2$ we have

$$\sum_{s=1}^{\omega(q_a)} (-1)^{s-1} \sum_{1\leq i_1<i_2......<i_s\leq \omega(q_a)} \frac{1}{\phi(a)} = \frac{a_{\omega(q_a)}}{\phi(a)}$$

$$\eta_{s,i} = \frac{(p_i-1)^{\delta_{p_i}(a)} \prod_{k=1}^{s} (p_{i_k}-1)^{\delta_{p_{i_k}}(a)}}{\phi(a) \prod_{k=1}^{s} (p_{i_k}-1)(p_i-1) \prod_{k=1}^{s} p_{i_k}^{\delta_{p_{i_k}}(a)} p_i^{\delta_{p_i}(a)}}$$

$$\mu(r) = \frac{1}{\phi(a)}[\sum_{s=1}^{\omega(q_a)} (-1)^{s-1} \sum_{1\leq i_1<i_2......<i_s\leq \omega(q_a)} \prod_{i=\omega(q_a)+1}^{r} (1-\eta_{s,i})]$$

Then

$$\frac{\beta(n)-\beta(\sqrt{n})}{\Pi(an+b)} = -\frac{2a_{\omega(q_a)}}{\phi(a)} + \frac{1}{\phi(a)} + \frac{4e^{-\gamma}c_2(n)\lambda_n(a)\epsilon(a)}{\phi(a)\ln n} + \bigcirc(\frac{1}{\ln^2 n}) - \mu(r), \forall a \not\equiv 0[2]$$

$$\frac{\beta(n)-\beta(\sqrt{n})}{\Pi(an+b)} = \frac{4e^{-\gamma}c_2(n)\lambda_n(a)\epsilon(a)}{\phi(a)\ln n} + \bigcirc(\frac{1}{\ln^2 n}) - \mu(r), \forall a \equiv 0[2]$$

3 Conclusion

In conclusion we retain that there is an infinity of prime numbers of Bado-Tiemoko. This result is a major step forward in the upcoming studies we will provide evidence of its generalization which will resolve Dickson's conjecture .

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