

# CATALAN MATRIX, BINOMIAL MATRIX, AND A NEWLY CONSTRUCTED ATALAN BINOMIAL INFINITE MATRIX METHOD WITH APPLICATION IN SUMMABILITY.

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## ABSTRACT

*Infinite matrices plays crucial roles in sequence and series spaces particularly in the summation of the divergent sequences and series. This paper explores the construction of a new matrix based summability method using the Catalan and the Binomial Matrices as the Base Matrix for its generation. The inverse of the Catalan and the Binomial Matrices were Obtained Under some imposed conditions and later on were characterized and applied to summability of some divergent sequences and series.*

*The inverse of newly constructed infinite matrix was obtain and later on was characterized and applied to the summation of some the summation of some divergent sequences and series. Further, we presented every application alongside with its geometric view to clearly justify our results.*

**Keywords:** Catalan Number, Catalan Matrix, Binomial Matrix, Catalan Binomial Matrix, summability, summability methods.

Mathematics Subject Classification: 15A09, 11B65, 15A16

## 1. Introduction

The pursuit of assigning meaningful values to divergent sequences and series, known as summability theory, stands as a mathematical ingenuity's testament. Mathematicians have since the pioneering work of Euler, Abel, Cesàro, and Borel, developed a vast arsenal of methods to extend the conventional notion of convergence. These methods are not merely mathematical inquisitiveness; they are indispensable tools in theoretical physics, engineering, and combinatorics, where divergent asymptotic expansions frequently arise and contain invaluable information about a system's behavior.

The concept **infinite matrix**  $A = (a_{nk})$  which defines a linear transformation from one sequence space into another new sequence space lies at the heart of many summability methods. If the new sequence  $\{t_n\}$ , where  $t_n = \sum_{k=1}^{\infty} a_{nk} x_k$  converges, then, the original series is said to be summable by the method  $A = (a_{nk})$ . The efficacy, strength, and regularity of any method are thus encoded in the properties of its defining matrix  $A = (a_{nk})$ .

Among the pantheon of infinite matrices, two, the Binomial Matrix ( $B^{r,s}$ ) and the Catalan Matrix ( $\check{C}$ ) hold a place of particular significance due to their profound combinatorial roots.

Here, we present their inverse, characterization, and application in summability under certain conditions. Further, we generated new infinite matrix summability method alongside with their inverse using both. Also, we characterized the new infinite matrix and applied it to summability theorem. This newly constructed infinite matrix is not merely a direct sum of its predecessors but a true hybrid, engineered to leverage the strengths of both worlds. In addition, the new matrix strictly stronger than both it's parental methods, capable of summing series that are intractable to both standard Borel and Catalan summability. This new method does not emerges just as a new mathematical object, but as a key that unlocks a deeper understanding of divergence, offering a more unified and powerful approach to

one of analysis's most enduring challenges.

## 2.0 Preliminaries

**Definition 2.1 (Catalan Number ( $C_n$ )):** Catalan number is a sequence of numbers given and defined by  $C_n = \frac{(2n)!}{n!(n+1)!}$  for all  $n \in \mathbb{N}$ . This implies that for  $n = 1, 2, 3, 4, 5, \dots$ , we have  $(C_n)_{n=0}^\infty = 1, 1, 2, 5, 14, 42, 132, 429, \dots$  which is also equivalent to  $C_{n+1} = \sum_{k=0}^n C_k C_{n-k}$ ,  $C_0 = C_1 = 1$

**Definition 2.2 (Catalan Matrix ( $\check{C}$ )):** This is defined by  $\check{C} = (\check{c}_{nk})$  where  $\check{c}_{nk} = \begin{cases} \frac{C_k C_{n-k}}{C_{n+1}}, & 0 \leq k \leq n \\ 0, & k \geq n \end{cases}$

using the Catalan numbers  $C_n = \frac{(2n)!}{n!(n+1)!}$  for all  $n \in \mathbb{N}$  which enumerate a vast array of combinatorial structures from valid parenthesis expressions to binary trees.

**Definition 2.3 (Binomial Matrix ( $B^{r,s}$ )):** The binomial matrix is defined by  $B^{r,s} = (b_{nk}^{r,s})$  where

$$b_{nk}^{r,s} = \begin{cases} \frac{1}{(s+r)^n} \binom{n}{k} s^{n-k} r^k, & 0 \leq k \leq n \\ 0, & k \geq n \end{cases} \quad \forall k, n \in \mathbb{N} \text{ and } r, s \text{ are non zero real numbers with } r + s \neq 0.$$

**Definition 2.4 (Stefan Stanimirović, 2009):** The Catalan matrix of order  $n$ , denoted by

$$I_n(x) = (C_{ij}(x)) \text{ is equal to } (C_{ij}(x)) = \begin{cases} x^{ij} C_{ij}, & i-j \geq 0 \\ 0, & i-j < 0 \end{cases} \quad i, j = 1, 2, 3, \dots, n \text{ where } C_k \text{ is the } k\text{-th}$$

Catalan number and  $x \in \mathbb{R}$ .

## 3.0 Some known results

**Corollary 3.1 (Nkuno DI\*, Kiltho A and Brono AM, 2025):** There exist a modulus operator  $\triangleleft_\gamma$  of order  $\gamma$  for every  $\alpha$ -circular porous bounded sequence  $\{x_n\}_{k=1}^\infty$  such that

$$(\xi_k)_{k=1}^\infty = (\triangleleft_\gamma x_i)_{i=0}^\infty = \left\{ \frac{\gamma i + 1}{\gamma i}, \text{ for each } i \right\},$$

$$\xi_n < \xi_{n+1} < \xi_{n+2} < \xi_{n+3} < \xi_{n+4} < \dots \text{ and } \lim_{n \rightarrow \infty} \xi_n \rightarrow \infty.$$

**Theorem 3.1 (Robert Mann, 1952):** Suppose  $A = a_{nk}$  is an infinite real matrix satisfying:

a.  $a_{nk} \geq 0$  for all  $n, j$ , and  $a_{nk} = 0$  for  $j > n$ ;

b.  $\sum_{k=0}^\infty a_{nk} = 1$  for all  $n$ ;

c.  $\lim_{n \rightarrow \infty} a_{nk} = 0$  for all  $j$ .

i.  $x_1 \in \mathbb{C}$

ii.  $v_n = \sum_{k=0}^\infty a_{nk} x_k$

iii.  $x_{n+1} = T v_n, n = 1, 2, 3, \dots$

Now, if either of the sequences  $(x_n)$  and  $(v_n)$  converges, then the other also converges to the same point, and their common limit is a fixpoint of  $T$ .

**Theorem 3.2 (Rhoades, 1974):** Let  $A$  be a regular weighted mean method satisfying

$$\lim_{n \rightarrow \infty} \sum_{k=0}^\infty |a_{n,k} - a_{n-1,k}| = 0 \text{ and let } f \text{ be a continuous mapping from } E = [a, b] \text{ into itself.}$$

Then the iteration scheme

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- i.  $x_0 = x_0 \in E$
  - ii.  $x_{n+1} = f(x_n), n \geq 0,$
  - iii.  $x_n = \sum_{k=0}^n a_{n,k} x_k, n \geq 1$  converges to a fixed point of  $f$ .

**Theorem 3.3 (Rhoades, 1974):** Let  $a, b \in \mathbb{R}, a < b, E = [a, b], f: E \rightarrow E, f$  continuous and with at most one fixed point. With  $A$  satisfying

$$a_{nk} = \begin{cases} c_k \sum_{j=k+1}^n (1-c_n), & k < n \\ c_n, & k = n \\ 0, & k > n \end{cases}$$

and with  $(c_n)$  real sequence satisfying

- (i)  $c_0 = 1$
- (ii)  $0 < c_n < 1$  for  $n \geq 1,$  and
- (iii)  $\sum_k c_k$  diverges and  $\lim_n c_n = 0,$

The iteration scheme,  $x_{n+1} = (1-c_n)x_n + c_n f(x_n),$  converges to the fixed point of  $f$ .

**Theorem 3.4 (Dotson, 1970):** Suppose  $E$  is a real Hilbert space,  $C$  is a closed convex subset of  $E, T: C \rightarrow C$  is quasi-nonexpansive on  $C$  and has at least one fixed point  $p \in C,$  and  $I-T$  is demiclosed. Suppose  $x_1 \in C$  and  $M(x_1, A, T)$  is a normal Mann process such that  $(t_n) = (a_{n+1, n+1})$  is bounded away from 0 and 1. Then the sequences  $(x_n), (v_n)$  converge weakly to a fixed point of  $T$ .

**Theorem 3.5 (Siddiqui and Nkuno, (2020)):** Let  $A = (a_{nk})$  be any infinite matrix and  $S$  be the space of all sequences of real or complex numbers. Then, the convergence field  $F(A)$  is porous in  $S(A)$  if and only if the matrix  $A = (a_{nk})$  is irregular.

**Theorem 3.6 (Nkuno DI\*, Kiltho A and Brono AM, 2025):** Let  $\{x_k\}_{k=1}^\infty$  be an  $\alpha$ -porous bounded sequence and  $\triangle_y$  be the modulus operator defined in **corollary 3.1** above. Then, there exist a Riesz matrix

$$R_p = (R_p) = \begin{cases} \frac{1}{n+1} & \text{if } k \leq n \\ 0, & \text{otherwise} \end{cases} \quad (n, k \in \mathbb{N}), \text{ corresponding to } \{x_k\}_{k=1}^\infty \text{ if and only if there exist a}$$

shift operator,  $\Delta$  such that  $(\Delta(\triangle_y x_k))_{n=1}^\infty = (1_n)_{n=1}^\infty$

## 4.0 Main results

**Discussion 4.1 (Catalan Matrix):** Here, we consider the special case of the Catalan matrix given in definition 2.4 above and defined by  $\check{C} = (\check{c}_{nk})$  Where

$$\check{c}_{nk} = \begin{cases} \frac{c_k c_{n-k}}{c_{n+1}}, & 1 \leq k \leq n \\ 0, & k \geq n \end{cases} \quad \text{and } c_k \text{ is a Catalan number given in definition 2.3 above.}$$

Thus, we have

$$\check{C} = \begin{pmatrix} \frac{c_0 c_0}{c_1} & 0 & 0 & 0 & 0 & 0 & 0 & \dots & \dots \\ \frac{c_0 c_1}{c_2} & \frac{c_1 c_0}{c_2} & 0 & 0 & 0 & 0 & 0 & 0 & \dots \\ \frac{c_0 c_2}{c_3} & \frac{c_1 c_1}{c_3} & \frac{c_2 c_0}{c_3} & 0 & 0 & 0 & 0 & \dots & \dots \\ \frac{c_0 c_3}{c_4} & \frac{c_1 c_2}{c_4} & \frac{c_2 c_1}{c_4} & \frac{c_3 c_0}{c_4} & 0 & 0 & 0 & \dots & \dots \\ \frac{c_0 c_4}{c_5} & \frac{c_1 c_3}{c_5} & \frac{c_2 c_2}{c_5} & \frac{c_3 c_1}{c_5} & \frac{c_4 c_0}{c_5} & 0 & 0 & \dots & \dots \\ \frac{c_0 c_5}{c_6} & \frac{c_1 c_4}{c_6} & \frac{c_2 c_3}{c_6} & \frac{c_3 c_2}{c_6} & \frac{c_4 c_1}{c_6} & \frac{c_5 c_0}{c_6} & 0 & \dots & \dots \\ \frac{c_0 c_6}{c_7} & \frac{c_1 c_5}{c_7} & \frac{c_2 c_4}{c_7} & \frac{c_3 c_3}{c_7} & \frac{c_4 c_2}{c_7} & \frac{c_5 c_1}{c_7} & \frac{c_6 c_0}{c_7} & \dots & \dots \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots \end{pmatrix}$$

$$\check{C} = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & \dots & \dots \\ \frac{1}{2} & \frac{1}{2} & 0 & 0 & 0 & 0 & 0 & \dots & \dots \\ \frac{2}{5} & \frac{1}{5} & \frac{2}{5} & 0 & 0 & 0 & 0 & \dots & \dots \\ \frac{5}{14} & \frac{2}{14} & \frac{2}{14} & \frac{5}{14} & 0 & 0 & 0 & \dots & \dots \\ \frac{14}{42} & \frac{5}{42} & \frac{4}{42} & \frac{5}{42} & \frac{14}{42} & 0 & 0 & \dots & \dots \\ \frac{42}{132} & \frac{14}{132} & \frac{10}{132} & \frac{10}{132} & \frac{14}{132} & \frac{42}{132} & 0 & \dots & \dots \\ \frac{132}{429} & \frac{42}{429} & \frac{28}{429} & \frac{25}{429} & \frac{28}{429} & \frac{42}{429} & \frac{132}{429} & \dots & \dots \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots \end{pmatrix}$$



### The Inverse of Catalan Matrix

**Theorem 4.1:** The Catalan matrix given above has the inverse  $\check{C} = \check{C}^{-1} = (\check{c}_{nk})^{-1}$  given by

$$\check{C}^{-1} = \begin{pmatrix} c_1 & 0 & 0 & 0 & 0 & 0 & \dots & \dots & \dots \\ -c_1 & c_2 & 0 & 0 & 0 & 0 & \dots & \dots & \dots \\ -\frac{c_1}{c_2} & -c_1 & \frac{c_3}{c_2} & 0 & 0 & 0 & \dots & \dots & \dots \\ -\frac{c_2}{c_3} & -\frac{c_2}{c_3} & -c_1 & \frac{c_4}{c_3} & 0 & 0 & \dots & \dots & \dots \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots \\ -\frac{c_3}{c_4} & -\frac{c_2}{c_3+c_2} & -\frac{c_3}{c_4} & -c_1 & c_2+c_1 & 0 & \dots & \dots & \dots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \end{pmatrix}$$

**Proof**

$$\begin{pmatrix} \frac{c_0 c_0}{c_1} & 0 & 0 & 0 & 0 & 0 & 0 & \dots & \dots & \dots \\ \frac{c_0 c_1}{c_2} & \frac{c_1 c_0}{c_2} & 0 & 0 & 0 & 0 & 0 & 0 & \dots & \dots \\ \frac{c_0 c_2}{c_3} & \frac{c_1 c_1}{c_3} & \frac{c_2 c_0}{c_3} & 0 & 0 & 0 & 0 & \dots & \dots & \dots \\ \frac{c_0 c_3}{c_4} & \frac{c_1 c_2}{c_4} & \frac{c_2 c_1}{c_4} & \frac{c_3 c_0}{c_4} & 0 & 0 & 0 & \dots & \dots & \dots \\ \frac{c_0 c_4}{c_5} & \frac{c_1 c_3}{c_5} & \frac{c_2 c_2}{c_5} & \frac{c_3 c_1}{c_5} & \frac{c_4 c_0}{c_5} & 0 & 0 & \dots & \dots & \dots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \end{pmatrix} \check{C}^{-1} = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & \dots & \dots & \dots \\ 0 & 1 & 0 & 0 & 0 & \dots & \dots & \dots \\ 0 & 0 & 1 & 0 & 0 & \dots & \dots & \dots \\ 0 & 0 & 0 & 1 & 0 & \dots & \dots & \dots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \end{pmatrix}$$

$$\begin{pmatrix} c_0 & 0 & 0 & 0 & 0 & \cdot & \cdot & \cdot \\ c_1 & c_1 & 0 & 0 & 0 & \cdot & \cdot & \cdot \\ c_2 & c_1 & c_2 & 0 & 0 & \cdot & \cdot & \cdot \\ c_3 & c_2^2 & c_3 & c_0 & 0 & \cdot & \cdot & \cdot \\ c_4 & c_3 & c_2^2 & c_3 & c_4 & \cdot & \cdot & \cdot \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \end{pmatrix} \tilde{C}^i = \begin{pmatrix} c_1 & 0 & 0 & 0 & 0 & \cdot & \cdot & \cdot \\ 0 & c_2 & 0 & 0 & 0 & \cdot & \cdot & \cdot \\ 0 & 0 & c_3 & 0 & 0 & \cdot & \cdot & \cdot \\ 0 & 0 & 0 & c_4 & 0 & \cdot & \cdot & \cdot \\ 0 & 0 & 0 & 0 & c_5 & \cdot & \cdot & \cdot \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \end{pmatrix}$$

$$\begin{pmatrix} 1 & 0 & 0 & 0 & 0 & \cdot & \cdot & \cdot \\ 1 & 1 & 0 & 0 & 0 & \cdot & \cdot & \cdot \\ 1 & \frac{c_1}{c_2} & 1 & 0 & 0 & \cdot & \cdot & \cdot \\ 1 & \frac{c_2}{c_3} & \frac{c_2}{c_3} & 1 & 0 & \cdot & \cdot & \cdot \\ 1 & \frac{c_3}{c_4} & \frac{c_2^2}{c_4} & \frac{c_3}{c_4} & 1 & \cdot & \cdot & \cdot \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \end{pmatrix} \tilde{C}^i = \begin{pmatrix} c_1 & 0 & 0 & 0 & 0 & \cdot & \cdot & \cdot \\ 0 & c_2 & 0 & 0 & 0 & \cdot & \cdot & \cdot \\ 0 & 0 & \frac{c_3}{c_2} & 0 & 0 & \cdot & \cdot & \cdot \\ 0 & 0 & 0 & \frac{c_4}{c_3} & 0 & \cdot & \cdot & \cdot \\ 0 & 0 & 0 & 0 & \frac{c_5}{c_4} & \cdot & \cdot & \cdot \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \end{pmatrix}$$

$$\begin{pmatrix} 1 & 0 & 0 & 0 & 0 & \cdot & \cdot & \cdot \\ 0 & 1 & 0 & 0 & 0 & \cdot & \cdot & \cdot \\ 0 & \frac{-c_1}{c_2} & 1 & 0 & 0 & \cdot & \cdot & \cdot \\ 0 & \frac{-c_1}{c_2 c_3} & \frac{-(c_2+c_1)}{c_3} & 1 & 0 & \cdot & \cdot & \cdot \\ 0 & \frac{-(c_2+c_1)}{c_3 c_4} & \frac{-c_2^2}{c_3(c_3+c_2)} & \frac{-(c_2^2+c_3)}{c_4} & 1 & \cdot & \cdot & \cdot \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \end{pmatrix} \tilde{C}^i = \begin{pmatrix} c_1 & 0 & 0 & 0 & 0 & \cdot & \cdot & \cdot \\ -c_1 & c_2 & 0 & 0 & 0 & \cdot & \cdot & \cdot \\ 0 & -c_2 & \frac{c_3}{c_2} & 0 & 0 & \cdot & \cdot & \cdot \\ 0 & 0 & -\frac{c_3}{c_2} & \frac{c_4}{c_3} & 0 & \cdot & \cdot & \cdot \\ 0 & 0 & 0 & -\frac{c_4}{c_3} & \frac{c_5}{c_4} & \cdot & \cdot & \cdot \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \end{pmatrix}$$

$$\left( \begin{array}{cccccc} 1 & 0 & 0 & 0 & 0 & \cdot & \cdot & \cdot \\ 0 & 1 & 0 & 0 & 0 & \cdot & \cdot & \cdot \\ 0 & 1 & -c_2 & 0 & 0 & \cdot & \cdot & \cdot \\ 0 & 1 & c_3+c_1 & -c_2c_3 & 0 & \cdot & \cdot & \cdot \\ 0 & 1 & \frac{c_2^3}{c_2+c_1} & \frac{c_3(c_2^2+c_3)}{c_2+c_1} & \frac{-c_3c_4}{c_2+c_1} & \cdot & \cdot & \cdot \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \end{array} \right) \tilde{C}' = \left( \begin{array}{cccccc} c_1 & 0 & 0 & 0 & 0 & \cdot & \cdot & \cdot \\ -c_1 & c_2 & 0 & 0 & 0 & \cdot & \cdot & \cdot \\ 0 & c_2^2 & -c_3 & 0 & 0 & \cdot & \cdot & \cdot \\ 0 & 0 & c_3^2 & -c_2c_4 & 0 & \cdot & \cdot & \cdot \\ 0 & 0 & 0 & \frac{c_4^2}{c_2+c_1} & -c_3c_4 & \cdot & \cdot & \cdot \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \end{array} \right)$$

$$\left( \begin{array}{cccccc} 1 & 0 & 0 & 0 & 0 & \cdot & \cdot & \cdot \\ 0 & 1 & 0 & 0 & 0 & \cdot & \cdot & \cdot \\ 0 & 0 & -c_2 & 0 & 0 & \cdot & \cdot & \cdot \\ 0 & 0 & c_2^3 & -c_2c_3 & 0 & \cdot & \cdot & \cdot \\ 0 & 0 & \frac{-c_2c_3}{c_2+c_1} & c_3^2 & \frac{-c_3c_4}{c_2+c_1} & \cdot & \cdot & \cdot \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \end{array} \right) \tilde{C}' = \left( \begin{array}{cccccc} c_1 & 0 & 0 & 0 & 0 & \cdot & \cdot & \cdot \\ -c_1 & c_2 & 0 & 0 & 0 & \cdot & \cdot & \cdot \\ c_1 & c_2^2 & -c_3 & 0 & 0 & \cdot & \cdot & \cdot \\ 0 & -c_2^2 & c_2(c_4+c_1) & -c_2c_4 & 0 & \cdot & \cdot & \cdot \\ 0 & 0 & -c_3^2 & \frac{c_2^2c_3c_4}{c_2+c_1} & -c_3c_4 & \cdot & \cdot & \cdot \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \end{array} \right)$$

$$\left( \begin{array}{cccccc} 1 & 0 & 0 & 0 & 0 & \cdot & \cdot & \cdot \\ 0 & 1 & 0 & 0 & 0 & \cdot & \cdot & \cdot \\ 0 & 0 & 1 & 0 & 0 & \cdot & \cdot & \cdot \\ 0 & 0 & 1 & -\frac{c_3}{c_2^2} & 0 & \cdot & \cdot & \cdot \\ 0 & 0 & 1 & \frac{-(c_4+c_1)}{c_2} & c_3+c_2 & \cdot & \cdot & \cdot \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \end{array} \right) \tilde{C}' = \left( \begin{array}{cccccc} c_1 & 0 & 0 & 0 & 0 & \cdot & \cdot & \cdot \\ -c_1 & c_2 & 0 & 0 & 0 & \cdot & \cdot & \cdot \\ -\frac{c_1}{c_2} & -c_1 & \frac{c_3}{c_2} & 0 & 0 & \cdot & \cdot & \cdot \\ 0 & \frac{-c_1}{c_2} & \frac{-(c_4+c_1)}{c_2^2} & \frac{-(c_3+c_2)}{c_2} & 0 & \cdot & \cdot & \cdot \\ 0 & 0 & \frac{c_3(c_2+c_1)}{c_2} & -c_2c_4 & \frac{c_4(c_2+c_1)}{c_2} & \cdot & \cdot & \cdot \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \end{array} \right)$$

$$\begin{pmatrix} 1 & 0 & 0 & 0 & 0 & \dots & \dots \\ 0 & 1 & 0 & 0 & 0 & \dots & \dots \\ 0 & 0 & 1 & 0 & 0 & \dots & \dots \\ 0 & 0 & 0 & -\frac{c_3}{c_2^2} & 0 & \dots & \dots \\ 0 & 0 & 0 & \frac{-c_3}{c_2^2} & c_3+c_2 & \dots & \dots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \end{pmatrix} \hat{C}^x = \begin{pmatrix} c_1 & 0 & 0 & 0 & 0 & \dots & \dots \\ -c_1 & c_2 & 0 & 0 & 0 & \dots & \dots \\ -\frac{c_1}{c_2} & -c_1 & \frac{c_3}{c_2} & 0 & 0 & \dots & \dots \\ \frac{c_1}{c_2} & \frac{c_1}{c_2} & \frac{c_3}{c_2^2} & \frac{-(c_3+c_2)}{c_2} & 0 & \dots & \dots \\ 0 & \frac{c_1}{c_2} & \frac{(c_4+c_1)}{c_2^2} & \frac{-(c_3+c_2)^2}{c_2} & \frac{c_4(c_2+c_1)}{c_2} & \dots & \dots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \end{pmatrix}$$

$$\begin{pmatrix} 1 & 0 & 0 & 0 & 0 & \dots & \dots \\ 0 & 1 & 0 & 0 & 0 & \dots & \dots \\ 0 & 0 & 1 & 0 & 0 & \dots & \dots \\ 0 & 0 & 0 & 1 & 0 & \dots & \dots \\ 0 & 0 & 0 & 1 & \frac{-c_2 c_4}{c_3^2} & \dots & \dots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \end{pmatrix} \hat{C}^x = \begin{pmatrix} c_1 & 0 & 0 & 0 & 0 & \dots & \dots \\ -c_1 & c_2 & 0 & 0 & 0 & \dots & \dots \\ -\frac{c_1}{c_2} & -c_1 & \frac{c_3}{c_2} & 0 & 0 & \dots & \dots \\ -\frac{c_2}{c_3} & -\frac{c_2}{c_3} & -c_1 & \frac{c_4}{c_3} & 0 & \dots & \dots \\ 0 & \frac{-c_2}{c_3^2} & \frac{-(c_2+c_1)}{c_3} & \frac{c_2 c_5 + c_4}{c_3^2} & \frac{-c_2 c_5}{c_3^2} & \dots & \dots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \end{pmatrix}$$

$$\begin{pmatrix} 1 & 0 & 0 & 0 & 0 & \dots & \dots \\ 0 & 1 & 0 & 0 & 0 & \dots & \dots \\ 0 & 0 & 1 & 0 & 0 & \dots & \dots \\ 0 & 0 & 0 & 1 & 0 & \dots & \dots \\ 0 & 0 & 0 & 0 & \frac{c_2 c_4}{c_3^2} & \dots & \dots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \end{pmatrix} \hat{C}^x = \begin{pmatrix} c_1 & 0 & 0 & 0 & 0 & \dots & \dots \\ -c_1 & c_2 & 0 & 0 & 0 & \dots & \dots \\ -\frac{c_1}{c_2} & -c_1 & \frac{c_3}{c_2} & 0 & 0 & \dots & \dots \\ -\frac{c_2}{c_3} & -\frac{c_2}{c_3} & -c_1 & \frac{c_4}{c_3} & 0 & \dots & \dots \\ \frac{c_2}{c_3} & \frac{c_2}{c_3} & \frac{c_2 c_4}{c_3^2} & \frac{-c_2 c_5}{c_3^2} & \dots & \dots & \dots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \end{pmatrix}$$

$$\begin{pmatrix} 1 & 0 & 0 & 0 & 0 & \cdot & \cdot & \cdot \\ 0 & 1 & 0 & 0 & 0 & \cdot & \cdot & \cdot \\ 0 & 0 & 1 & 0 & 0 & \cdot & \cdot & \cdot \\ 0 & 0 & 0 & 1 & 0 & \cdot & \cdot & \cdot \\ 0 & 0 & 0 & 0 & 1 & \cdot & \cdot & \cdot \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \end{pmatrix} \check{C}^i = \begin{pmatrix} c_1 & 0 & 0 & 0 & 0 & \cdot & \cdot & \cdot \\ -c_1 & c_2 & 0 & 0 & 0 & \cdot & \cdot & \cdot \\ -\frac{c_1}{c_2} & -c_1 & \frac{c_3}{c_2} & 0 & 0 & \cdot & \cdot & \cdot \\ \frac{c_2}{c_3} & -\frac{c_2}{c_3} & -c_1 & \frac{c_4}{c_3} & 0 & \cdot & \cdot & \cdot \\ \frac{-c_3}{c_4} & \frac{-c_2}{c_3+c_2} & \frac{-c_3}{c_4} & -c_1 & c_2+c_1 & 0 & \cdot & \cdot \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \end{pmatrix}$$

$$\check{C}^i = \begin{pmatrix} c_1 & 0 & 0 & 0 & 0 & 0 & \cdot & \cdot & \cdot \\ -c_1 & c_2 & 0 & 0 & 0 & 0 & \cdot & \cdot & \cdot \\ -\frac{c_1}{c_2} & -c_1 & \frac{c_3}{c_2} & 0 & 0 & 0 & \cdot & \cdot & \cdot \\ \frac{c_2}{c_3} & -\frac{c_2}{c_3} & -c_1 & \frac{c_4}{c_3} & 0 & 0 & \cdot & \cdot & \cdot \\ \frac{-c_3}{c_4} & \frac{-c_2}{c_3+c_2} & \frac{-c_3}{c_4} & -c_1 & c_2+c_1 & 0 & \cdot & \cdot & \cdot \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \end{pmatrix}$$

Which is now the inverse of the Catalan matrix given and defined by  $\check{C} = (\check{c}_{nk})$  where

$$\check{C} = \begin{cases} \frac{c_k c_{n-k}}{c_{n+1}}, & 0 \leq k < n \\ \frac{c_k c_{n-k}}{c_{n+1}}, & k = n \\ 0, & k \geq n \end{cases}$$

Which now proves the result.

**Lemma 4.1:** Let  $C_n$  be the Catalan number, then, the Catalan matrix  $\check{C} = (\check{c}_{nk})$  defined above is regular.

**Proof:**

Suppose that  $\check{C} = (\check{c}_{nk})$  is Toeplitz matrix. Then,

i.  $\sum_{k=1}^{\infty} |\check{c}_{nk}| = \sum_{k=1}^{\infty} \left| \frac{c_k c_{n-k}}{c_{n+1}} \right| = \left| \frac{1}{c_{n+1}} \sum_{k=1}^{\infty} c_k c_{n-k} \right| = \left| \frac{R_m}{c_{n+1}} \right| = M < \infty$   
 $\Rightarrow \sum_{k=1}^{\infty} |\check{c}_{nk}| = M < \infty$  for each  $n = 1, 2, \dots$  fulfilling condition (i) of Toeplitz matrix

ii.  $\lim_{n \rightarrow \infty} \check{c}_{nk} = \lim_{n \rightarrow \infty} \frac{1}{c_{n+1}} \sum_{k=1}^{\infty} c_k c_{n-k} = \lim_{n \rightarrow \infty} \frac{R_m}{c_{n+1}}$   
 $= R_m \lim_{n \rightarrow \infty} \frac{1}{c_{n+1}} = R_m \cdot 0 = 0$   
 $\Rightarrow \lim_{n \rightarrow \infty} \check{c}_{nk} = 0$  (this is condition (i) of the Toeplitz matrix)

iii.  $\lim_{n \rightarrow \infty} \sum_{k=1}^{\infty} \check{c}_{nk} = \lim_{n \rightarrow \infty} \sum_{k=1}^{\infty} \frac{c_k c_{n-k}}{c_{n+1}} = \lim_{n \rightarrow \infty} \frac{1}{c_{n+1}} \sum_{k=1}^{\infty} c_k c_{n-k} = \lim_{n \rightarrow \infty} \frac{1}{c_{n+1}} \cdot c_{n+1} = \lim_{n \rightarrow \infty} \frac{c_{n+1}}{c_{n+1}}$   
 $= \lim_{n \rightarrow \infty} (1) = 1$

Since  $c_{n+1} = \sum_{k=1}^n c_k c_{n-k}$ ,

$\Rightarrow \lim_{n \rightarrow \infty} \sum_{k=1}^{\infty} \check{c}_{nk} = 1$  (Hm, this is nothing but the Toeplitz third condition for the regularity)

**Example C1:** Now, if  $(x_k) = (1, 0, 1, 0, 1, 0, 1, \dots)$

Then, we know that

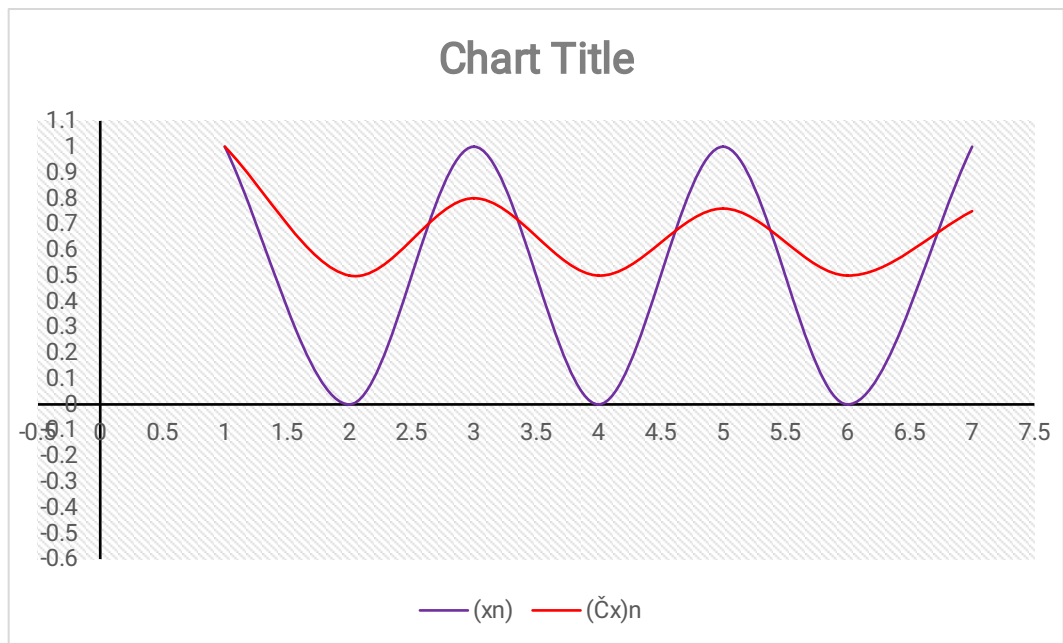
$$\lim_{k \rightarrow \infty} x_k = 0$$

But then,

$$(\check{C}x)_n = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & \dots \\ \frac{1}{2} & \frac{1}{2} & 0 & 0 & 0 & 0 & 0 & \dots \\ \frac{2}{5} & \frac{1}{5} & \frac{2}{5} & 0 & 0 & 0 & 0 & \dots \\ \frac{5}{14} & \frac{2}{14} & \frac{2}{14} & \frac{5}{14} & 0 & 0 & 0 & \dots \\ \frac{14}{42} & \frac{5}{42} & \frac{4}{42} & \frac{5}{42} & \frac{14}{42} & 0 & 0 & \dots \\ \frac{42}{132} & \frac{14}{132} & \frac{10}{132} & \frac{10}{132} & \frac{14}{132} & \frac{42}{132} & 0 & \dots \\ \frac{132}{429} & \frac{42}{429} & \frac{28}{429} & \frac{25}{429} & \frac{28}{429} & \frac{42}{429} & \frac{132}{429} & \dots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix} = \begin{pmatrix} 1+0+\dots \\ \frac{1}{2}+0+\dots \\ \frac{2}{5}+0+\frac{2}{5}+0+\dots \\ \frac{5}{14}+0+\frac{2}{14}+0 \\ \frac{14}{42}+0+\frac{4}{42}+0+\frac{14}{42}+0+\dots \\ \frac{42}{132}+0+\frac{10}{132}+0+\frac{14}{132}+0+ \\ \frac{132}{429}+0+\frac{28}{429}+0+\frac{28}{429}+0+\frac{132}{429} \\ \vdots \end{pmatrix} = \begin{pmatrix} 1 \\ \frac{1}{2} \\ \frac{4}{5} \\ \frac{1}{2} \\ \frac{16}{21} \\ \frac{33}{66} \\ \frac{320}{429} \\ \vdots \end{pmatrix}$$

$$\Rightarrow (\check{C}x)_n = \begin{pmatrix} 1 \\ \frac{1}{2} \\ \frac{4}{5} \\ \frac{1}{2} \\ \frac{16}{21} \\ \frac{1}{2} \\ \frac{320}{429} \\ \vdots \end{pmatrix}$$

Clearly,  $\lim_{n \rightarrow \infty} (\check{C}x)_n \rightarrow 1$



**Example C2:** Let  $x_n = \left(\frac{1}{n}\right) \forall n \in \mathbb{N}$ . Then,

We know that  $\lim_{n \rightarrow \infty} x_n = \lim_{n \rightarrow \infty} \frac{1}{n} \rightarrow 0$

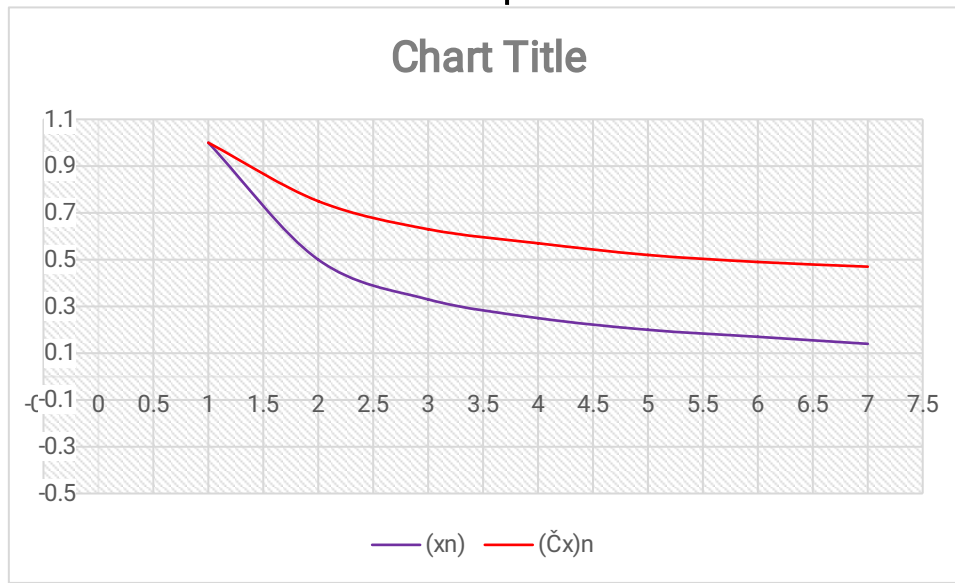
Further,

$$\begin{aligned}
 (\check{C}x) &= \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \dots \\ \frac{1}{2} & \frac{1}{2} & 0 & 0 & 0 & 0 & 0 & 0 & \dots \\ \frac{2}{5} & \frac{1}{5} & \frac{2}{5} & 0 & 0 & 0 & 0 & 0 & \dots \\ \frac{5}{14} & \frac{2}{14} & \frac{2}{14} & \frac{5}{14} & 0 & 0 & 0 & 0 & \dots \\ \frac{14}{42} & \frac{5}{42} & \frac{4}{42} & \frac{5}{42} & \frac{14}{42} & 0 & 0 & 0 & \dots \\ \frac{42}{132} & \frac{14}{132} & \frac{10}{132} & \frac{10}{132} & \frac{14}{132} & \frac{42}{132} & 0 & 0 & \dots \\ \frac{132}{429} & \frac{42}{429} & \frac{28}{429} & \frac{25}{429} & \frac{28}{429} & \frac{42}{429} & \frac{132}{429} & \dots & \dots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \end{pmatrix} \begin{pmatrix} 1 \\ \frac{1}{2} \\ \frac{1}{3} \\ \frac{1}{4} \\ \frac{1}{5} \\ \frac{1}{6} \\ \frac{1}{7} \\ \vdots \\ \vdots \end{pmatrix} \\
 &= \begin{pmatrix} 1+0+\dots \\ \frac{1}{2}+\frac{1}{4}+0+\dots \\ \frac{2}{5}+\frac{1}{10}+\frac{2}{15}+0+\dots \\ \frac{5}{14}+\frac{1}{14}+\frac{1}{21}+\frac{5}{56}+0+\dots \\ \frac{1}{3}+\frac{5}{84}+\frac{2}{63}+\frac{5}{168}+\frac{1}{15}+0+\dots \\ \frac{7}{22}+\frac{7}{132}+\frac{5}{198}+\frac{5}{264}+\frac{7}{330}+\frac{7}{132}+\dots \\ \frac{132}{429}+\frac{42}{858}+\frac{28}{1287}+\frac{25}{1716}+\frac{28}{2145}+\frac{42}{2574}+\frac{132}{3003}+\dots \\ \vdots \\ \vdots \end{pmatrix} = \begin{pmatrix} 1 \\ \frac{3}{4} \\ \frac{19}{30} \\ \frac{95}{168} \\ \frac{1313}{2520} \\ \frac{21329}{43560} \\ \frac{84017}{180180} \\ \vdots \\ \vdots \end{pmatrix}
 \end{aligned}$$

$$\Rightarrow (\check{C}x) = \begin{pmatrix} 1 \\ 3 \\ 4 \\ \frac{19}{30} \\ \frac{1313}{2520} \\ \vdots \end{pmatrix}$$

Clearly,  $\lim_{n \rightarrow \infty} (\check{C}x)_n \rightarrow 0$

### Geometrical representation



**Example C3:** Let  $x_n = \left(\frac{1}{2}\right)^n \forall n \in \mathbb{N}$  be a sequence. Then  $\lim_{n \rightarrow \infty} \left(\frac{1}{2}\right)^n \rightarrow 0$

Thus,

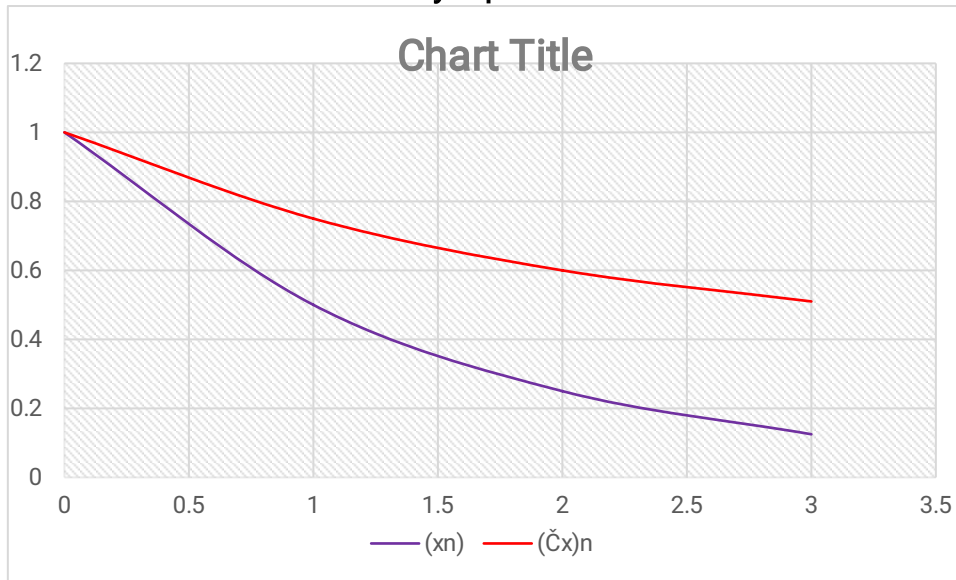
$$(\check{C}x)_n = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & \dots \\ \frac{1}{2} & \frac{1}{2} & 0 & 0 & 0 & 0 & 0 & \dots \\ \frac{2}{5} & \frac{1}{5} & \frac{2}{5} & 0 & 0 & 0 & 0 & \dots \\ \frac{5}{14} & \frac{2}{14} & \frac{2}{14} & \frac{5}{14} & 0 & 0 & 0 & \dots \\ \frac{14}{42} & \frac{5}{42} & \frac{4}{42} & \frac{5}{42} & \frac{14}{42} & 0 & 0 & \dots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix} \begin{pmatrix} 1 \\ \frac{1}{2} \\ \frac{1}{4} \\ \frac{1}{8} \\ \frac{1}{16} \\ \vdots \end{pmatrix} = \begin{pmatrix} 1+0+\dots \\ \frac{1}{2}+\frac{1}{4}+0+\dots \\ \frac{2}{5}+\frac{1}{10}+\frac{2}{20}+0+\dots \\ \frac{5}{14}+\frac{2}{28}+\frac{2}{56}+\frac{5}{112}+0+\dots \\ \frac{14}{42}+\frac{5}{84}+\frac{4}{168}+\frac{5}{336}+\frac{14}{672}+0+\dots \\ \vdots \end{pmatrix}$$

$$(\check{C}x)_n = \begin{pmatrix} 1 \\ 3 \\ 4 \\ \frac{3}{5} \\ \frac{57}{112} \\ \vdots \end{pmatrix}$$

$$\Rightarrow (\check{C}x)_n = \begin{pmatrix} 1 \\ \frac{3}{4} \\ \frac{3}{5} \\ \frac{57}{112} \\ \vdots \\ \vdots \end{pmatrix}$$

Clearly,  $\lim_{n \rightarrow \infty} (\check{C}x)_n \rightarrow 0$

### Geometry representation



**Example C4:** Let  $x_n = \frac{2n^2+1}{3n^2+7} \forall n \in \mathbb{N}^0$  be a sequence. Then  $\lim_{n \rightarrow \infty} \frac{2n^2+1}{3n^2+7} \rightarrow \frac{2}{3}$

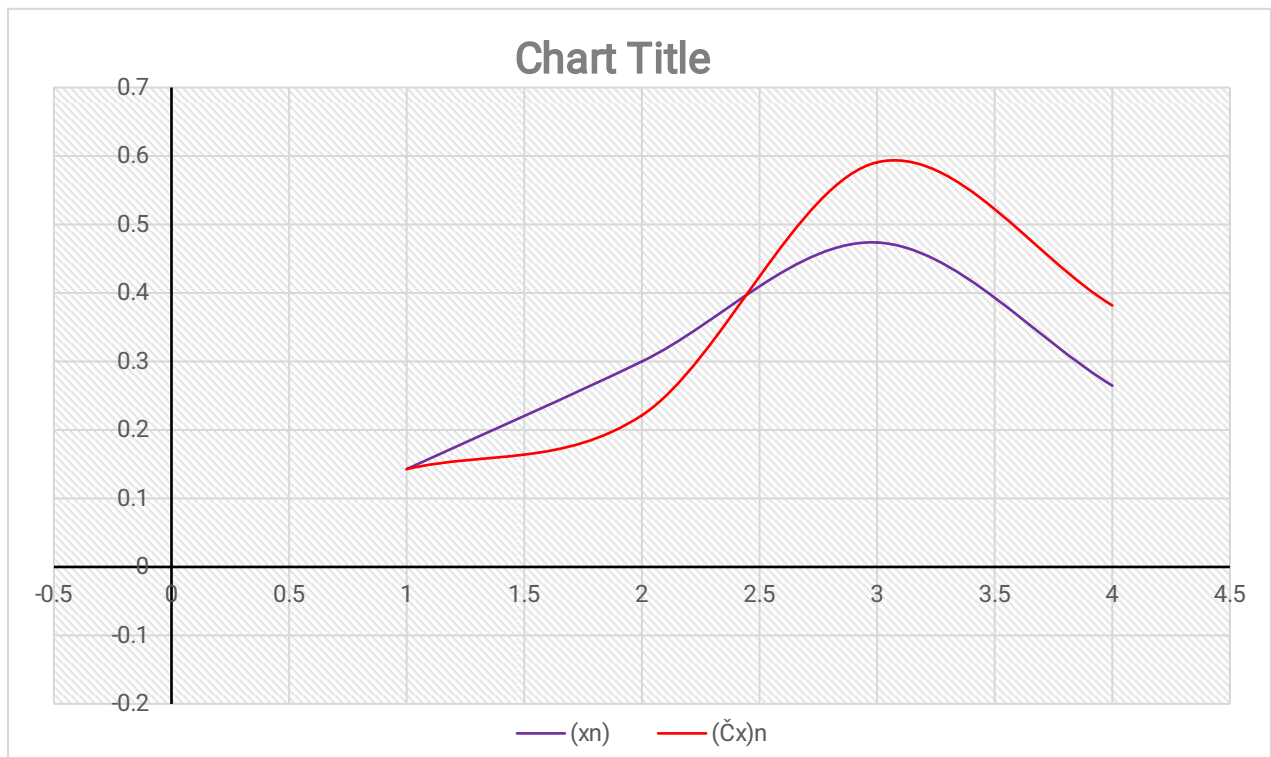
$$(\check{C}x) = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & \dots \\ \frac{1}{2} & \frac{1}{2} & 0 & 0 & 0 & 0 & 0 & \dots \\ \frac{2}{5} & \frac{1}{5} & \frac{2}{5} & 0 & 0 & 0 & 0 & \dots \\ \frac{5}{14} & \frac{2}{14} & \frac{2}{14} & \frac{5}{14} & 0 & 0 & 0 & \dots \\ \frac{14}{42} & \frac{5}{42} & \frac{4}{42} & \frac{5}{42} & \frac{14}{42} & 0 & 0 & \dots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \end{pmatrix} \begin{pmatrix} \frac{1}{7} \\ \frac{3}{10} \\ \frac{9}{19} \\ \frac{19}{34} \\ \frac{32}{55} \\ \vdots \\ \vdots \end{pmatrix}$$

$$\begin{pmatrix} \frac{1}{7} + 0 + \dots \\ \frac{1}{14} + \frac{3}{20} + 0 + \dots \\ \frac{2}{35} + \frac{3}{50} + \frac{18}{95} + 0 + \dots \\ \frac{5}{98} + \frac{6}{140} + \frac{18}{261} + \frac{195}{476} + 0 + \dots \\ \frac{14}{294} + \frac{35}{420} + \frac{36}{821} + \frac{95}{1428} + \frac{32}{165} + 0 + \dots \\ \vdots \end{pmatrix} = \begin{pmatrix} \frac{1}{7} \\ \frac{31}{140} \\ \frac{3929}{6650} \\ \frac{1936207}{5072970} \\ \vdots \end{pmatrix}$$

$$\Rightarrow (\check{C}x) = \begin{pmatrix} \frac{1}{7} \\ \frac{31}{140} \\ \frac{3929}{6650} \\ \frac{1936207}{5072970} \\ \vdots \end{pmatrix}$$

Clearly,  $\lim_{n \rightarrow \infty} (\check{C}x)_n \rightarrow 0$

### Geometry representation



#### Definition (Observation):

A sequence is a mapping between two set/spaces in which each element of the set/space is mapped to the element of the other set/space e.g. by the sequence  $x_n = \frac{1}{n} \quad \forall n \in \mathbb{N}$ , we mean a mapping  $f: \mathbb{N} \rightarrow [0,1]$ .

Thus, by  $s$  the space of all sequences we simply mean the space of all mapping  $f$  for which the image(s) exist in the given codomain for each of the element in the domain.

**Theorem 4.2:** Let  $s$  be the space of all sequences and let  $\check{C} = (\check{c}_{nk})$  be the Catalan matrix defined above. Let  $D_{\check{C}}$  and  $C_{\check{C}}$  be the application and the convergence domain of  $\check{C} = (\check{c}_{nk})$  under the mapping  $f$  defined by

$$f(x) = \sum_{k=1}^{\infty} \check{c}_{nk} x_k.$$

Then,

- the convergence domain  $C_{\check{C}}$  of  $\check{C} = (\check{c}_{nk})$  is fixed in the application domain  $D_{\check{C}}$  of  $\check{C} = (\check{c}_{nk})$  if and only if  $\check{C} = (\check{c}_{nk})$  is conservatively regular
- the spaces  $C_{\check{C}}^0$ ,  $C_{\check{C}}^1$ , and  $C_{\check{C}}$  is an extension field of the spaces  $C_0$ ,  $C_1$ , and  $C$  if and only if  $\check{C} = (\check{c}_{nk})$  is conservatively regular.

**Prof:**

- suppose that  $\check{C} = (\check{c}_{nk})$  conservatively regular. Then, for every  $x \in D_{\check{C}}$  there exist a mapping  $f$  such that  $f(x) = \sum_{k=1}^{\infty} \check{c}_{nk} x_k$  exist.

Further, there exist a mapping  $g: D_{\check{C}} \rightarrow C_{\check{C}}$  such that for all  $\forall x \in C_{\check{C}}$

$$g(f(x)) = g\left(\sum_{k=1}^{\infty} \check{c}_{nk} x_k\right).$$

But then,

$$\lim_{n \rightarrow \infty} (f(x)) = \lim_{n \rightarrow \infty} \left( \sum_{k=1}^{\infty} \check{c}_{nk} x_k \right) \quad \forall x \in C_{\check{c}}$$

Also, we have that

$$\sum_{k=1}^{\infty} \check{c}_{nk} x_k = t_n \quad \forall x \in D_{\check{c}}$$

$$\Rightarrow \lim_{n \rightarrow \infty} (f(x)) = \lim_{n \rightarrow \infty} \sum_{k=1}^{\infty} \check{c}_{nk} x_k = \lim_{n \rightarrow \infty} t_n = t \quad \forall x \in C_{\check{c}}$$

$$\Rightarrow \lim_{n \rightarrow \infty} (f(x)) = t$$

But then,

$$\check{C} = (\check{c}_{nk}) \text{ is regular.}$$

Thus by **theorem 3.6** above,

$$\lim_{n \rightarrow \infty} (f(x)) = \lim_{n \rightarrow \infty} \sum_{k=1}^{\infty} \check{c}_{nk} x_k = \lim_{n \rightarrow \infty} t_n = t \quad \forall x \in D_{\check{c}}$$

Again,  $\check{C} = (\check{c}_{nk})$  is conservative.

Hence,

$$\lim_{n \rightarrow \infty} (f(x)) = \lim_{n \rightarrow \infty} (g(f(x))) = t \quad \forall x \in C_{\check{c}} D_{\check{c}}$$

$$\Rightarrow f(x) = g(f(x))$$

$$\Rightarrow g^{-1}(f(x)) = f(x)$$

$$\Rightarrow g^{-1}f(x) = f(x)$$

$$\Rightarrow g^{-1}f = f$$

$$\Rightarrow g^{-1} = 1$$

$$\Rightarrow g = 1$$

$$\Rightarrow g^{-1} = g$$

This implies that  $g$  is one to one and onto and therefore, the convergence domain  $C_{\check{c}}$  is fixed in the application domain  $D_{\check{c}}$  of the Catalan matrix  $\check{C} = (\check{c}_{nk})$ .

Conversely, suppose that, the convergence domain  $C_{\check{c}}$  is fixed in the application domain  $D_{\check{c}}$  of the Catalan matrix  $\check{C} = (\check{c}_{nk})$ . Furthermore, let  $G(C_{\check{c}})$  and  $G(D_{\check{c}})$  be the graph of the convergence domain and the application domain of the Catalan matrix  $\check{C} = (\check{c}_{nk})$ . Then,

$$G(C_{\check{c}}) \cap G(D_{\check{c}}) \neq \emptyset$$

but then,  $C_{\check{c}} \subset D_{\check{c}}$  from **theorem 3.6** above.

Therefore, every mapping  $f: D_{\check{c}} \rightarrow C_{\check{c}}$  is self-mapping. Further, since  $C_{\check{c}}$  is dense in  $D_{\check{c}}$  by **theorem 3.6** above, it follows that for every  $x \in D_{\check{c}}$

$$\lim_{n \rightarrow \infty} (f(x)) = \lim_{n \rightarrow \infty} \left( \sum_{k=1}^{\infty} \check{c}_{nk} x_k \right) = t \quad \forall x \in D_{\check{c}}$$

But then,

$$\Rightarrow \lim_{n \rightarrow \infty} (f(x)) = \lim_{n \rightarrow \infty} \sum_{k=1}^{\infty} \check{c}_{nk} x_k = t \quad \forall x \in C_{\check{c}}$$

This implies that  $f$  is one to one and the Catalan matrix  $\check{C} = (\check{c}_{nk})$  is conservative.

ii. Suppose that  $\check{C} = (\check{c}_{nk})$  is conservative. Then,

$$\lim_{k \rightarrow \infty} x_k = \lim_{n \rightarrow \infty} t_n$$

This implies that if

$$\lim_{k \rightarrow \infty} x_k = 0 \quad \forall x \in C_0$$

$$\lim_{k \rightarrow \infty} x_k = 1 \quad \forall x \in C_1 \text{ and}$$

$$\lim_{k \rightarrow \infty} x_k = t \quad \forall x \in C$$

then,

$$\lim_{n \rightarrow \infty} t_n = 0 \quad \forall x \in C_{\check{c}}^0$$

$$\lim_{n \rightarrow \infty} t_n = 1 \quad \forall x \in C_{\check{c}}^1 \text{ and}$$

$$\lim_{n \rightarrow \infty} t_n = t \quad \forall x \in C_{\check{c}}$$

But then, the Catalan Matrix  $\check{C} = (\check{c}_{nk})$  is regular. Thus by **example C<sub>1</sub>** and **C<sub>4</sub>** above, there exist some sequence  $x = (x_k) \in s$  such that

$$\lim_{k \rightarrow \infty} x_k \neq 0 \text{ or } \lim_{k \rightarrow \infty} x_k = \infty$$

But however,

$$\lim_{n \rightarrow \infty} (\check{C}x)_n = \lim_{n \rightarrow \infty} \left( \sum_{k=1}^{\infty} \check{c}_{nk} x_k \right) = \lim_{n \rightarrow \infty} t_n = t$$

Now, if  $t = 0$ , then,  $(\check{C}x)_n \in C_{\check{c}}^0$  but  $x_k \notin C_0$

$$\Rightarrow C_0 \subset C_{\check{c}}^0$$

In the same way, if  $t = 1$ ,  $(\check{C}x)_n \in C_{\check{c}}^1$  but  $x_k \notin C_1$  which implies that  $C_1 \subset C_{\check{c}}^1$  and finally, if  $t = t$ , then,  $(\check{C}x)_n \in C_{\check{c}}$  but  $x_k \notin C_{\check{c}}$  which implies that  $C \subset C_{\check{c}}$ .

Thus it is obviously clear that the convergence domain  $C_{\check{c}}^0$ ,  $C_{\check{c}}^1$ , and  $C_{\check{c}}$  of the Catalan matrix  $\check{C} = (\check{c}_{nk})$  is the extension field of the corresponding sequences spaces  $C_0^0$ ,  $C_1^1$ , and  $C$  respectively.

Conversely, suppose that the convergence domain  $C_{\check{c}}^0$ ,  $C_{\check{c}}^1$ , and  $C_{\check{c}}$  of the Catalan matrix  $\check{C} = (\check{c}_{nk})$  are the

extension field of the corresponding sequences spaces  $C_0$ ,  $C_1$ , and  $C$ . Then, it will follow that there exist some sequence  $x = (x_k) \in s$  such that  $(\check{C}x)_n \in C_{\check{C}}^0$  but  $x_k \notin C_0$ . This implies that

$$\lim_{n \rightarrow \infty} (\check{C}x)_n = t = 0$$

Now, by deduction, there exist some sequence  $x = (x_k) \in s$  such that  $(\check{C}x)_n \in C_{\check{C}}^1$  but  $x_k \notin C_1$  and finally, there exist some sequence  $x = (x_k) \in s$  such,  $(\check{C}x)_n \in C_{\check{C}}$  but  $x_k \notin C_{\check{C}}$ . But then, the Catalan matrix  $\check{C} = (\check{c}_{nk})$  is a lower triangular Toeplitz matrix.

Therefore,  $\forall x = (x_k) \in C_0$ ,  $(\check{C}x)_n \in C_{\check{C}}^0$ .

This implies that

$$\lim_{n \rightarrow \infty} (\check{C}x)_n = t = 0$$

Furthermore, for all  $x = (x_k) \in C_1$ ,  $(\check{C}x)_n \in C_{\check{C}}^1$  and finally,  $x = (x_k) \in C_{\check{C}}$ ,  $(\check{C}x)_n \in C_{\check{C}}$ .

This implies that the matrix  $\check{C} = (\check{c}_{nk})$  is conservatively regular and hence the result.

**Example Cs:** Let  $x_n = \frac{n+1}{n} \forall n \in \mathbb{N}$ ,  $\beta < n$  be a sequence. Then  $\lim_{n \rightarrow \infty} x_n \rightarrow 1$

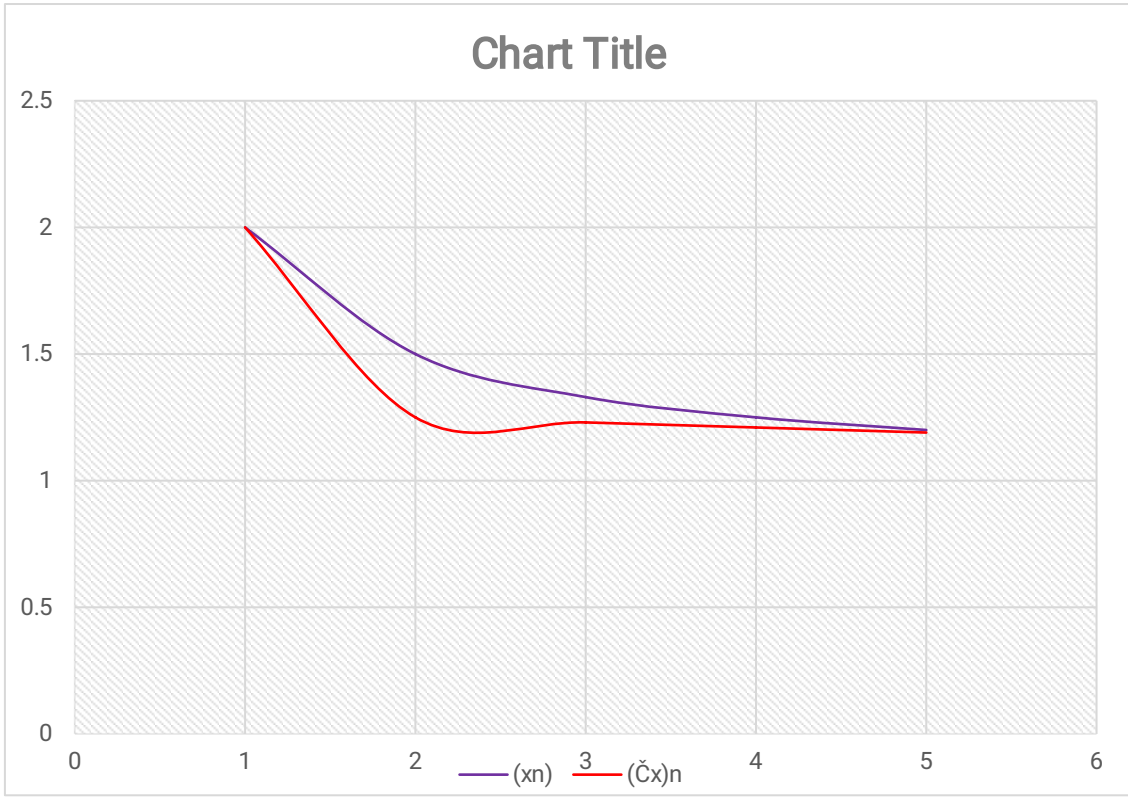
Further,

$$(\check{C}x) = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \dots \\ \frac{1}{2} & \frac{1}{2} & 0 & 0 & 0 & 0 & 0 & 0 & \dots \\ \frac{2}{5} & \frac{1}{5} & \frac{2}{5} & 0 & 0 & 0 & 0 & 0 & \dots \\ \frac{5}{14} & \frac{2}{14} & \frac{2}{14} & \frac{5}{14} & 0 & 0 & 0 & 0 & \dots \\ \frac{14}{42} & \frac{5}{42} & \frac{4}{42} & \frac{5}{42} & \frac{14}{42} & 0 & 0 & 0 & \dots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix} \begin{pmatrix} 2 \\ \frac{3}{2} \\ 4 \\ \frac{3}{5} \\ 5 \\ \frac{4}{4} \\ 6 \\ \frac{5}{5} \\ 7 \\ \frac{6}{6} \\ \vdots \\ \vdots \end{pmatrix}$$

$$\begin{pmatrix} 2+0+\dots \\ \frac{1}{2}+\frac{3}{4}+0+\dots \\ \frac{2}{5}+\frac{3}{10}+\frac{8}{15}+0+\dots \\ \frac{5}{14}+\frac{6}{28}+\frac{8}{42}+\frac{25}{56}+0+\dots \\ \frac{14}{42}+\frac{15}{84}+\frac{16}{126}+\frac{25}{168}+\frac{84}{210}+0+\dots \\ \vdots \end{pmatrix} = \begin{pmatrix} 2 \\ \frac{5}{4} \\ \frac{37}{30} \\ \frac{203}{168} \\ \frac{2999}{2520} \\ \vdots \end{pmatrix} = \begin{pmatrix} 2 \\ 1+\frac{1}{4} \\ 1+\frac{7}{30} \\ 1+\frac{35}{168} \\ 1+\frac{479}{2520} \\ \vdots \end{pmatrix}$$

$$\Rightarrow (\check{C}x) = \begin{pmatrix} 1 \\ 1+\frac{1}{4} \\ 1+\frac{7}{30} \\ 1+\frac{35}{168} \\ 1+\frac{479}{2520} \\ \vdots \\ \vdots \end{pmatrix}$$

Clearly,  $\lim_{n \rightarrow \infty} (\check{C}x)_n \rightarrow 1$



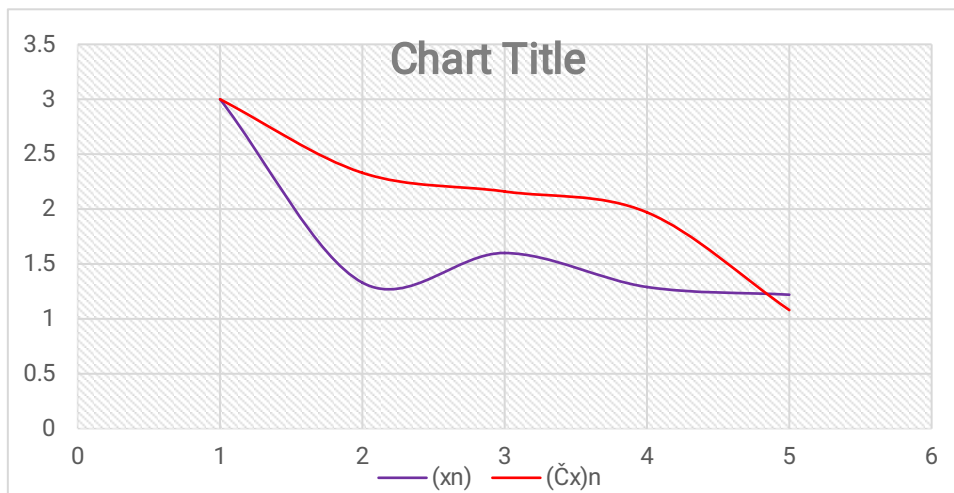
**Example C6:** Let  $x_n = \frac{2n+1}{2n-1} \forall n \in \mathbb{N}$  be a sequence. Then  $\lim_{n \rightarrow \infty} x_n = 1$

Further,

$$\begin{aligned}
 (\check{C}x) &= \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & \dots \\ \frac{1}{2} & \frac{1}{2} & 0 & 0 & 0 & 0 & 0 & \dots \\ \frac{2}{5} & \frac{1}{5} & \frac{2}{5} & 0 & 0 & 0 & 0 & \dots \\ \frac{5}{14} & \frac{2}{14} & \frac{2}{14} & \frac{5}{14} & 0 & 0 & 0 & \dots \\ \frac{14}{42} & \frac{5}{42} & \frac{4}{42} & \frac{5}{42} & \frac{14}{42} & 0 & 0 & \dots \\ \frac{132}{429} & \frac{42}{429} & \frac{28}{429} & \frac{25}{429} & \frac{28}{429} & \frac{42}{429} & \frac{132}{429} & \dots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix} \begin{pmatrix} 3 \\ \frac{5}{3} \\ \frac{7}{5} \\ \frac{9}{7} \\ \frac{11}{9} \\ \frac{13}{11} \\ \frac{15}{13} \\ \vdots \end{pmatrix} = \begin{pmatrix} 3+0+\dots \\ \frac{3}{2}+\frac{5}{6}+0+\dots \\ \frac{6}{5}+\frac{5}{15}+\frac{14}{25}+0+\dots \\ \frac{15}{14}+\frac{10}{42}+\frac{14}{70}+\frac{45}{98}+0+\dots \\ \frac{42}{42}+\frac{25}{126}+\frac{28}{210}+\frac{45}{294}+\frac{154}{378}+0+\dots \\ \vdots \\ \vdots \end{pmatrix} \\
 &= \begin{pmatrix} 3 \\ \frac{7}{3} \\ \frac{54}{25} \\ \frac{1447}{735} \\ \frac{71,862}{39690} \\ \vdots \\ \vdots \end{pmatrix} = \begin{pmatrix} 3 \\ 2+\frac{1}{3} \\ 2+\frac{4}{25} \\ 1+\frac{712}{735} \\ 1+\frac{562}{6615} \\ \vdots \\ \vdots \end{pmatrix}
 \end{aligned}$$

$$\Rightarrow (\check{C}x) = \begin{pmatrix} 3 \\ 2 + \frac{1}{3} \\ 2 + \frac{4}{25} \\ 1 + \frac{712}{735} \\ 1 + \frac{562}{6615} \\ \vdots \\ \vdots \end{pmatrix}$$

Clearly,  $\lim_{n \rightarrow \infty} (\check{C}x)_n \rightarrow 0$



**Theorem 4.3:** Let  $s$  be the space of all sequences and let  $\check{C} = (\check{c}_{nk})$  be the Catalan matrix defined above and let  $D_{\check{C}}$  and  $C_{\check{C}}$  be the application domain and the convergence domain of  $\check{C} = (\check{c}_{nk})$  under the mapping  $f$  defined by

$$f(x) = \sum_{k=1}^{\infty} \check{c}_{nk} x_k.$$

Then,:

- The first term of the sequence  $x = (x_k)$  is fixed under  $f$  if and only if  $\check{C} = (\check{c}_{nk})$  is a lower triangular matrix.
- $f$  Converges to one of its fixed point (if and only if  $\check{C} = (\check{c}_{nk})$  is conservatively regular).
- The spaces  $C_0$ ,  $C_1$ , and  $C$  is fixed in the corresponding convergence domain  $C_{\check{C}}^0$ ,  $C_{\check{C}}^1$ , and  $C_{\check{C}}$  of the matrix  $\check{C} = (\check{c}_{nk})$  if and only if  $\check{C} = (\check{c}_{nk})$  is conservatively regular.

**Prof:**

- Suppose that the first term of  $x = (x_k)$  is fixed under  $f$ . Then, it will follow that each  $x = (x_k) \in s$  such that

$$f(x) = \sum_{k=1}^{\infty} \check{c}_{nk} x_k \text{ exist, we have}$$

$$f(x_1) = x_1$$

But then, we know that

$$f(x_1) = \sum_{k=1}^1 \check{c}_{n1k} x_1$$

$$\Rightarrow \sum_{k=1}^1 \check{c}_{n1k} x_1 = x_1$$

$$\Rightarrow \check{c}_{nk} x_1 = x_1$$

$$\Rightarrow \check{c}_{nk} = 1$$

But also,  $\check{C} = (\check{c}_{nk})$  is a lower triangular. Therefore,

$$\sum_{k=1}^{\infty} \check{c}_{nk} = 1$$

$$\Rightarrow \lim_{k \rightarrow \infty} \sum_{k=1}^{\infty} \check{c}_{nk} = 1$$

In addition,

$$\lim_{k \rightarrow \infty} \check{c}_{nk} \rightarrow 0 \text{ and}$$

$$\sum_{k=1}^{\infty} \check{c}_{nk} = M < \infty$$

Which implies that  $\check{C} = (\check{c}_{nk})$  is a Toeplitz matrix and therefore regular.

Conversely, suppose that  $\check{C} = (\check{c}_{nk})$  is a Toeplitz matrix. Then, it will follow from **theorem T** that

a.  $\sup (\sum_{k=1}^{\infty} \check{c}_{nk}) = M < \infty$

b.  $\lim_{k \rightarrow \infty} \sum_{k=1}^{\infty} \check{c}_{nk} = 1$

c.  $\lim_{k \rightarrow \infty} \check{c}_{nk} = 0$

Now therefore, if  $x = (x_k) \in s$  is a sequence such that

$$f(x) = \sum_{k=1}^{\infty} \check{c}_{nk} x_k$$

Then, it will follow that

$$f(x_1) = \sum_{k=1}^n \check{c}_{01} x_1$$

But then,  $\check{C} = (\check{c}_{nk})$  is a lower triangular matrix

$$\check{c}_{11} = 1 \text{ and } \check{c}_{1k} = 0 \quad \forall k > 1$$

$$\Rightarrow f(x_1) = \check{c}_{11} x_1 = 1 \cdot x_1 = x_1$$

$$\Rightarrow f(x_1) = x_1$$

ii. Suppose that  $f$  Converges to one of its fixed term. Then, it will follow that

$$\lim_{k \rightarrow \infty} f(x) = x_k^*, \quad k = 1, 2, 3, 4, \dots$$

But however,

$$f(x) = \sum_{k=1}^{\infty} \check{c}_{nk} x_k$$

$$\Rightarrow \lim_{n \rightarrow \infty} f(x) = \lim_{n \rightarrow \infty} (\sum_{k=1}^{\infty} \check{c}_{nk} x_k) = x_k^*, \quad k = 1, 2, 3, 4, \dots$$

Also,

$$f(x_k^*) = \sum_{k=1}^{\infty} \check{c}_{nk} x_k^* = x_k^*$$

$$\Rightarrow \lim_{n \rightarrow \infty} f(x_k^*) = \lim_{n \rightarrow \infty} (\sum_{k=1}^{\infty} \check{c}_{nk} x_k^*) = \lim_{n \rightarrow \infty} x_k^* = x_k^*$$

Further, we know that  $\check{C} = (\check{c}_{nk})$  is a lower triangular matrix. Therefore,

$$\Rightarrow \lim_{n \rightarrow \infty} f(x_k^*) = \lim_{n \rightarrow \infty} (\sum_{k=1}^n \check{c}_{nk} x_k^*) = \lim_{n \rightarrow \infty} x_k^* = x_k^*$$

$$\Rightarrow \lim_{n \rightarrow \infty} f(x_k^*) = x_k^*$$

But then,  $\check{C} = (\check{c}_{nk})$  is a Toeplitz lower triangular matrix and therefore

a.  $\lim_{k \rightarrow \infty} \check{c}_{nk} = 0$

b.  $\sup (\sum_{k=1}^{\infty} \check{c}_{nk}) = M < \infty$

c.  $\lim_{k \rightarrow \infty} \sum_{k=1}^{\infty} \check{c}_{nk} = 1$

In addition, we know that if  $x_k$  is a fixed term of  $f$ , then, it will follow that

$$f(x_k) = x_k \text{ and } \lim_{k \rightarrow \infty} f(x_k) = \lim_{k \rightarrow \infty} x_k = x_k^*$$

$$\Rightarrow \lim_{n \rightarrow \infty} f(x_k) = x_k^*$$

Thus, we have

$$\Rightarrow \lim_{n \rightarrow \infty} f(x_k^*) = \lim_{n \rightarrow \infty} f(x_k)$$

$$\Rightarrow \lim_{n \rightarrow \infty} x_k^* = \lim_{n \rightarrow \infty} f(x_k)$$

Thus the matrix  $\check{C} = (\check{c}_{nk})$  is conservatively regular.

Conversely, suppose that the matrix  $\check{C} = (\check{c}_{nk})$  is conservatively. Then, it will follow that for any sequence

$(x_k) \in s$  such that

$$f(x) = \sum_{k=1}^{\infty} \check{c}_{nk} x_k$$

We have

$$\Rightarrow \lim_{n \rightarrow \infty} f(x_k) = \lim_{n \rightarrow \infty} x_k$$

$$\Rightarrow \lim_{n \rightarrow \infty} \sum_{k=1}^{\infty} \check{c}_{nk} x_k = \lim_{n \rightarrow \infty} x_k$$

$$\Rightarrow \sum_{k=1}^{\infty} \check{c}_{nk} x_k = x_k$$

But then,

$$f(x) = \sum_{k=1}^{\infty} \check{c}_{nk} x_k$$

$$\Rightarrow f(x) = x_k$$

$$\Rightarrow f(x_k) = x_k$$

This implies that the term  $x_k$  of the sequence  $x = (x_k)$  is fixed. Furthermore, suppose that

$$\lim_{n \rightarrow \infty} f(x_k) = \lim_{n \rightarrow \infty} x_k = x_k^*$$

$$\lim_{n \rightarrow \infty} f(x) = x_k^*$$

This implies that  $f$  converges to at least one of its fixed point and hence the result.

iii. Supposed that the matrix  $\check{C} = (\check{c}_{nk})$  is conservatively regular. Then, it will follow from **Theorem C<sub>1</sub> ii** above that

$$C_0 \subset C_{\check{C}}^0, \quad C_1 \subset C_{\check{C}}^1, \text{ and } C \subset C_{\check{C}}$$

$$\lim_{k \rightarrow \infty} x_k = \lim_{n \rightarrow \infty} t_n$$

This implies that since

$$\lim_{k \rightarrow \infty} x_k = 0 \quad \forall x \in C_0$$

$$\lim_{k \rightarrow \infty} x_k = 1 \quad \forall x \in C_1 \text{ and}$$

$$\lim_{k \rightarrow \infty} x_k = t \quad \forall x \in C$$

then,

$$\lim_{n \rightarrow \infty} t_n = 0 \quad \forall x \in C_{\check{c}}$$

$$\lim_{n \rightarrow \infty} t_n = 1 \quad \forall x \in C_{\check{c}}^1 \text{ and}$$

$$\lim_{n \rightarrow \infty} t_n = t \quad \forall x \in C_{\check{c}}$$

But then, the Catalan Matrix  $\check{C} = (\check{c}_{nk})$  is regular. Thus by **example C<sub>1</sub>** and **C<sub>4</sub>** above, there exist some sequence  $(x_k) \in s$  such that

$$\lim_{k \rightarrow \infty} x_k \neq 0 \text{ or } \lim_{k \rightarrow \infty} x_k = \infty$$

But however,

$$\lim_{n \rightarrow \infty} (\check{C}x)_n = \lim_{n \rightarrow \infty} (\sum_{k=1}^{\infty} \check{c}_{nk} x_k) = \lim_{n \rightarrow \infty} t_n = 0$$

$$(\check{C}x)_n \in C_{\check{c}}^0 \text{ but } x_k \notin C_0$$

$$\Rightarrow C_0 \subset C_{\check{c}}^0$$

Now since for each  $x = (x_k) \in C_{0'}$ ,

$$\lim_{k \rightarrow \infty} x_k = \lim_{n \rightarrow \infty} t_n$$

It follows that

$$f(x_k) = x_k$$

And therefore,

$$f(C_0) = C_0, \quad \forall x = (x_k) \in C_0$$

In the same way,  $\forall x = (x_k) \in C_1$  we have  $f(x_k) = x_k$

$$\Rightarrow f(C_1) = C_1, \quad \forall x = (x_k) \in C_1$$

Finally,  $\forall x = (x_k) \in C$  we have  $f(x_k) = x_k$

$$\Rightarrow f(C) = C, \quad \forall x = (x_k) \in C$$

Imply that  $C$  is fixed  $C_{\check{c}}$ .

Thus, it is obviously clear that the space  $C_{0'}$ ,  $C_1$ , and  $C$  is fixed in their respective corresponding convergence domain  $C_{\check{c}}^0$ ,  $C_{\check{c}}^1$  and  $C_{\check{c}}$  of the Catalan matrix  $\check{C} = (\check{c}_{nk})$ .

Conversely, suppose that the space  $C_{0'}$ ,  $C_1$ , and  $C$  are fixed in their respective corresponding convergence domain  $C_{\check{c}}^0$ ,  $C_{\check{c}}^1$  and  $C_{\check{c}}$  of the Catalan matrix  $\check{C} = (\check{c}_{nk})$ . Then, it will follow that

$$f(C_0) = C_0, \quad \forall x = (x_k) \in C_0, \quad f(C_1) = C_1, \quad \forall x = (x_k) \in C_1 \text{ and } f(C) = C, \quad \forall x = (x_k) \in C$$

$$\Rightarrow f(x) = x = (x_k)$$

$$f(x) = \sum_{k=1}^{\infty} \check{c}_{nk} x_k$$

$$\Rightarrow f(x) = \sum_{k=1}^{\infty} \check{c}_{nk} x_k = x_k$$

Now, we have

$$\lim_{n \rightarrow \infty} f(x) = \lim_{n \rightarrow \infty} \sum_{k=1}^{\infty} \check{c}_{nk} x_k = \lim_{n \rightarrow \infty} x_k$$

$$\Rightarrow \lim_{n \rightarrow \infty} f(x) = \lim_{n \rightarrow \infty} x_k$$

Implies that the Catalan matrix  $\check{C} = (\check{c}_{nk})$  is conservatively regular.

**Discussion 4.2 (Binomial matrix):** In the section, we consider the special case of the binomial matrix

$B^{r,s} = (b_{nk}^{r,s})$  with entries given and defined by

$$b_{nk}^{r,s} = \begin{cases} \frac{1}{(s+r)^n} \binom{n}{k} s^{n-k} r^k, & 0 \leq k \leq n \\ 0, & k > n \end{cases} \quad \forall k, n \in \mathbb{N} \text{ and } r, s \text{ are non zero real numbers with } r + s \neq 0$$

Thus, the special binomial matrix  $B^{r,s} = (b_{nk}^{r,s})$  can roosterly be written as follows

$$B^{r,s} = (b_{nk}^{r,s}) = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & \dots \\ \frac{s}{(r+s)} & \frac{r}{(r+s)} & 0 & 0 & 0 & 0 & \dots \\ \frac{s^2}{(r+s)^2} & \frac{2sr}{(r+s)^2} & \frac{r^2}{(r+s)^2} & 0 & 0 & 0 & \dots \\ \frac{s^3}{(r+s)^3} & \frac{3s^2r}{(r+s)^3} & \frac{3sr^2}{(r+s)^3} & \frac{r^3}{(r+s)^3} & 0 & 0 & \dots \\ \frac{s^4}{(r+s)^4} & \frac{4s^3r}{(r+s)^4} & \frac{6s^2r^2}{(r+s)^4} & \frac{4sr^3}{(r+s)^4} & \frac{r^4}{(r+s)^4} & 0 & \dots \\ \frac{s^5}{(r+s)^5} & \frac{5s^4r}{(r+s)^5} & \frac{10s^3r^2}{(r+s)^5} & \frac{10s^2r^3}{(r+s)^5} & \frac{5sr^4}{(r+s)^5} & \frac{r^5}{(r+s)^5} & \dots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \dots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \dots \end{pmatrix}$$

Now, if we set  $s = r \neq 0$ , then, it will follow that

$$B^{r,s} = (b_{nk}^{s,s}) = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & \dots \\ \frac{1}{2} & \frac{1}{2} & 0 & 0 & 0 & 0 & \dots \\ \frac{1}{4} & \frac{1}{2} & \frac{1}{4} & 0 & 0 & 0 & \dots \\ \frac{1}{8} & \frac{3}{8} & \frac{3}{8} & \frac{1}{8} & 0 & 0 & \dots \\ \frac{1}{16} & \frac{4}{16} & \frac{6}{16} & \frac{4}{16} & \frac{1}{16} & 0 & \dots \\ \frac{1}{32} & \frac{5}{32} & \frac{10}{32} & \frac{10}{32} & \frac{5}{32} & \frac{1}{32} & \dots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix} \quad \text{--- i}$$

When  $s < r$  (e.g  $s = 2, r = 3$ ), we have

$$B_{nk}^{3,2} = (b_{nk}^{2,2}) = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & \dots \\ \frac{2}{5} & \frac{3}{5} & 0 & 0 & 0 & 0 & 0 & \dots \\ \frac{4}{25} & \frac{12}{25} & \frac{9}{25} & 0 & 0 & 0 & 0 & \dots \\ \frac{8}{125} & \frac{36}{125} & \frac{54}{125} & \frac{27}{125} & 0 & 0 & 0 & \dots \\ \frac{16}{625} & \frac{96}{625} & \frac{216}{625} & \frac{216}{625} & \frac{81}{625} & 0 & 0 & \dots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix} \quad \text{--- ii}$$

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When  $s > r$  (e.g  $s = 7, r = 2$ ), then,

$$B^{r,s} = (b_{nk}^{2,7}) = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & \dots \\ \frac{7}{9} & \frac{2}{9} & 0 & 0 & 0 & 0 & \dots \\ \frac{49}{81} & \frac{28}{81} & \frac{4}{81} & 0 & 0 & 0 & \dots \\ \frac{343}{729} & \frac{294}{729} & \frac{84}{729} & \frac{8}{729} & 0 & 0 & \dots \\ \frac{2401}{6561} & \frac{2744}{6561} & \frac{1176}{6561} & \frac{224}{6561} & \frac{16}{6561} & 0 & \dots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix} \quad \text{--- iii}$$

When both  $s$  and  $r$  are fractionis  $s = \frac{2}{3}, r = \frac{3}{4}$ , then,

$$B^{\frac{3}{4}, \frac{2}{3}} = (b_{nk}^{r,s}) = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & \dots \\ \frac{8}{17} & \frac{9}{17} & 0 & 0 & 0 & 0 & \dots \\ \frac{64}{289} & \frac{144}{289} & \frac{81}{289} & 0 & 0 & 0 & \dots \\ \frac{512}{4913} & \frac{1728}{4913} & \frac{1944}{4913} & \frac{729}{4913} & 0 & 0 & \dots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix} \quad \text{--- iv}$$

when  $s$  is a whole number and  $r$  is a fraction (e. g  $s = 2, r = \frac{2}{5}$ ), then,

$$B_{5^s}^{\frac{2}{5}} = \left( b_{nk}^{\frac{2}{5}, 3} \right) = \left( b_{nk}^{r, s} \right) = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & \dots \\ \frac{15}{17} & \frac{2}{17} & 0 & 0 & 0 & 0 & \dots \\ \frac{225}{289} & \frac{60}{289} & \frac{4}{289} & 0 & 0 & 0 & \dots \\ \frac{40}{4913} & \frac{900}{4913} & \frac{1350}{4913} & \frac{3375}{4913} & 0 & 0 & \dots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \end{pmatrix} - v$$

when one of  $s$  and  $r$  is negative (e.g  $s = -1, r = 2$ ), then,

$$B^{r,s} = (b_{nk}^{2,-1}) = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & \dots & \dots \\ -1 & 2 & 0 & 0 & 0 & 0 & \dots & \dots \\ 1 & -4 & 4 & 0 & 0 & 0 & \dots & \dots \\ -1 & 6 & -12 & 8 & 0 & 0 & \dots & \dots \\ 1 & -8 & 24 & -32 & 16 & 0 & \dots & \dots \\ -1 & 10 & -40 & 80 & -80 & 32 & 0 & \dots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix} \quad \text{-- vi}$$

when  $s$  and  $r$  are congrate (e.g  $s = s, r = -s$ ), then,

$$B^{-S,S} = (b_{nk}^{-S,S}) = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & \dots & \dots \\ \infty & \infty & 0 & 0 & 0 & 0 & \dots & \dots \\ \infty & \infty & \infty & 0 & 0 & 0 & \dots & \dots \\ \infty & \infty & \infty & \infty & 0 & 0 & \dots & \dots \\ \infty & \infty & \infty & \infty & \infty & 0 & \dots & \dots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix} \quad \text{-- - - - -vii}$$

when both are negative (e.g  $s = -\frac{2}{3}, r = -\frac{3}{4}$ ), then,

$$B^{\frac{3}{4}, \frac{2}{3}} = (b_{nk}^{\frac{3}{4}, \frac{2}{3}}) = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & \dots & \dots \\ \frac{8}{17} & \frac{9}{17} & 0 & 0 & 0 & 0 & \dots & \dots \\ \frac{64}{289} & \frac{144}{289} & \frac{81}{289} & 0 & 0 & 0 & \dots & \dots \\ \frac{512}{4913} & \frac{1728}{4913} & \frac{1944}{4913} & \frac{729}{4913} & 0 & 0 & \dots & \dots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix} \quad \text{-- -viii}$$

**The inverse of binomial matrix**

**Theorem 4.4:** The Binomial matrix discussion 4.2 above is invertible provided that  $r \neq -s$

$$B^{r,s} = B^{r,s-1} = (b_{nk}^{r,s})^{-1}$$

$$B^{r,s} B^{r,s} = I$$

### Proof

$$\left( \begin{array}{cccccccc} 1 & 0 & 0 & 0 & 0 & 0 & \dots & \\ \frac{s}{(r+s)} & \frac{r}{(r+s)} & 0 & 0 & 0 & 0 & \dots & \\ \frac{s^2}{(r+s)^2} & \frac{2sr}{(r+s)^2} & \frac{r^2}{(r+s)^2} & 0 & 0 & 0 & \dots & \\ \frac{s^3}{(r+s)^3} & \frac{3s^2r}{(r+s)^3} & \frac{3sr^2}{(r+s)^3} & \frac{r^3}{(r+s)^3} & 0 & 0 & \dots & \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \end{array} \right) B^{r,s} = \left( \begin{array}{cccccccc} 1 & 0 & 0 & 0 & 0 & 0 & \dots & \\ 0 & 1 & 0 & 0 & 0 & 0 & \dots & \\ 0 & 0 & 1 & 0 & 0 & 0 & \dots & \\ 0 & 0 & 0 & 1 & 0 & 0 & \dots & \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \end{array} \right)$$

$$\Rightarrow \left( \begin{array}{cccccccc} 1 & 0 & 0 & 0 & 0 & 0 & \dots & \\ s & r & 0 & 0 & 0 & 0 & \dots & \\ s^2 & 2sr & r^2 & 0 & 0 & 0 & \dots & \\ s^3 & 3s^2r & 3sr^2 & r^3 & 0 & 0 & \dots & \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \end{array} \right) B^{r,s} = \left( \begin{array}{cccccccc} 1 & 0 & 0 & 0 & 0 & 0 & \dots & \\ 0 & \left(\frac{r}{s}+1\right) & 0 & 0 & 0 & 0 & \dots & \\ 0 & 0 & \left(\frac{r}{s}+1\right)^2 & 0 & 0 & 0 & \dots & \\ 0 & 0 & 0 & \left(\frac{r}{s}+1\right)^3 & 0 & 0 & \dots & \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \end{array} \right)$$

$$\Rightarrow \left( \begin{array}{cccccccc} 1 & 0 & 0 & 0 & 0 & 0 & \dots & \\ 1 & \frac{r}{s} & 0 & 0 & 0 & 0 & \dots & \\ 1 & \frac{2r}{s} & \left(\frac{r}{s}\right)^2 & 0 & 0 & 0 & \dots & \\ 1 & \frac{3r}{s} & \frac{3r^2}{s^2} & \left(\frac{r}{s}\right)^3 & 0 & 0 & \dots & \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \end{array} \right) B^{r,s} = \left( \begin{array}{cccccccc} 1 & 0 & 0 & 0 & 0 & 0 & \dots & \\ 0 & \left(\frac{r+s}{s}\right) & 0 & 0 & 0 & 0 & \dots & \\ 0 & 0 & \left(\frac{r+s}{s}\right)^2 & 0 & 0 & 0 & \dots & \\ 0 & 0 & 0 & \left(\frac{r+s}{s}\right)^3 & 0 & 0 & \dots & \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \end{array} \right)$$

$$\Rightarrow \left( \begin{array}{cccccccc} 1 & 0 & 0 & 0 & 0 & 0 & \dots & \\ 0 & \frac{r}{s} & 0 & 0 & 0 & 0 & \dots & \\ 0 & \frac{r}{s} & \left(\frac{r}{s}\right)^2 & 0 & 0 & 0 & \dots & \\ 0 & \frac{r}{s} & 2\left(\frac{r}{s}\right)^2 & \left(\frac{r}{s}\right)^3 & 0 & 0 & \dots & \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \end{array} \right) \mathbf{B}^{r,s} = \left( \begin{array}{cccccccc} 1 & 0 & 0 & 0 & 0 & 0 & \dots & \\ -1 & \left(\frac{r+s}{s}\right) & 0 & 0 & 0 & 0 & \dots & \\ 0 & -\left(\frac{r+s}{s}\right) & \left(\frac{r+s}{s}\right)^2 & 0 & 0 & 0 & \dots & \\ 0 & 0 & -\left(\frac{r+s}{s}\right)^2 & \left(\frac{r+s}{s}\right)^3 & 0 & 0 & \dots & \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \end{array} \right)$$

$$\Rightarrow \left( \begin{array}{cccccccc} 1 & 0 & 0 & 0 & 0 & 0 & \dots & \\ 0 & 1 & 0 & 0 & 0 & 0 & \dots & \\ 0 & 1 & \frac{r^2}{sr} & 0 & 0 & 0 & \dots & \\ 0 & 1 & 2\frac{r^2}{sr} & \frac{r^3}{s^2r} & 0 & 0 & \dots & \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \end{array} \right) \mathbf{B}^{r,s} = \left( \begin{array}{cccccccc} 1 & 0 & 0 & 0 & 0 & 0 & \dots & \\ -\frac{s}{r} & \left(\frac{r+s}{r}\right) & 0 & 0 & 0 & 0 & \dots & \\ 0 & -\left(\frac{r+s}{r}\right) & \frac{(r+s)^2}{sr} & 0 & 0 & 0 & \dots & \\ 0 & 0 & -\frac{(r+s)^2}{sr} & \frac{(r+s)^3}{s^2r} & 0 & 0 & \dots & \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \end{array} \right)$$

$$\Rightarrow \left( \begin{array}{cccccccc} 1 & 0 & 0 & 0 & 0 & 0 & \dots & \\ 0 & 1 & 0 & 0 & 0 & 0 & \dots & \\ 0 & 0 & \frac{r}{s} & 0 & 0 & 0 & \dots & \\ 0 & 0 & \frac{r}{s} & \frac{r^2}{s^2} & 0 & 0 & \dots & \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \end{array} \right) \mathbf{B}^{r,s} = \left( \begin{array}{cccccccc} 1 & 0 & 0 & 0 & 0 & 0 & \dots & \\ -\frac{s}{r} & \left(\frac{r+s}{r}\right) & 0 & 0 & 0 & 0 & \dots & \\ \frac{s}{r} & -2\left(\frac{r+s}{r}\right) & \frac{(r+s)^2}{sr} & 0 & 0 & 0 & \dots & \\ 0 & \left(\frac{r+s}{r}\right) & -\frac{(r+s)^2}{sr} & \frac{(r+s)^3}{s^2r} & 0 & 0 & \dots & \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \end{array} \right)$$

$$\Rightarrow \left( \begin{array}{cccccccc} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \dots \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & \dots \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & \dots \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & \dots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{array} \right) B^{r,s} = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & \dots \\ -\frac{s}{r} & \left(\frac{r+s}{r}\right) & 0 & 0 & 0 & 0 & \dots \\ \left(\frac{s}{r}\right)^2 & -2s\left(\frac{r+s}{r^2}\right) & \frac{(r+s)^2}{r^2} & 0 & 0 & 0 & \dots \\ 0 & s\left(\frac{r+s}{r^2}\right) & -\frac{(r+s)^2}{r^2} & \frac{(r+s)^3}{sr^2} & 0 & 0 & \dots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}$$

$$\Rightarrow \left( \begin{array}{cccccccc} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \dots \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & \dots \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & \dots \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & \dots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{array} \right) B^{r,s} = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & \dots \\ -\frac{s}{r} & \left(\frac{r+s}{r}\right) & 0 & 0 & 0 & 0 & \dots \\ \left(\frac{s}{r}\right)^2 & -2s\left(\frac{r+s}{r^2}\right) & \left(\frac{r+s}{r}\right)^2 & 0 & 0 & 0 & \dots \\ \left(\frac{s}{r}\right)^2 & 3s\left(\frac{r+s}{r^2}\right) & -2\left(\frac{r+s}{r}\right)^2 & \frac{(r+s)^3}{sr^2} & 0 & 0 & \dots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}$$

$$\Rightarrow \left( \begin{array}{cccccccc} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \dots \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & \dots \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & \dots \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & \dots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{array} \right) B^{r,s} = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & \dots \\ -\frac{s}{r} & \left(\frac{r+s}{r}\right) & 0 & 0 & 0 & 0 & \dots \\ \left(\frac{s}{r}\right)^2 & -2s\left(\frac{r+s}{r^2}\right) & \left(\frac{r+s}{r}\right)^2 & 0 & 0 & 0 & \dots \\ \left(\frac{s}{r}\right)^4 & 3s^3\left(\frac{r+s}{r^3}\right) & -2s^2\left(\frac{r+s}{r^2}\right)^2 & \frac{s(r+s)^3}{r^4} & 0 & 0 & \dots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}$$

$$\Rightarrow B^{r,s} = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & \dots \\ -\frac{s}{r} & \left(\frac{r+s}{r}\right) & 0 & 0 & 0 & 0 & \dots \\ \left(\frac{s}{r}\right)^2 & -2s\left(\frac{r+s}{r^2}\right) & \left(\frac{r+s}{r}\right)^2 & 0 & 0 & 0 & \dots \\ \left(\frac{s}{r}\right)^4 & 3s^3\left(\frac{r+s}{r^3}\right) & -2s^2\left(\frac{r+s}{r^2}\right)^2 & \frac{s(r+s)^3}{r^4} & 0 & 0 & \dots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}$$

Thus the binomial matrix  $B^{r,s} = (b_{nk}^{r,s})$  has the inverse  $B^{r,s} = B^{r,s-1} = (b_{nk}^{r,s})^{-1}$  as

$$B^{r,s} = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & \dots \\ -\frac{s}{r} & \left(\frac{r+s}{r}\right) & 0 & 0 & 0 & 0 & \dots \\ \left(\frac{s}{r}\right)^2 & -2s\left(\frac{r+s}{r^2}\right) & \left(\frac{r+s}{r}\right)^2 & 0 & 0 & 0 & \dots \\ \left(\frac{s}{r}\right)^4 & 3s^3\left(\frac{r+s}{r^3}\right) & -2s^2\left(\frac{r+s}{r^2}\right)^2 & \frac{s(r+s)^3}{r^4} & 0 & 0 & \dots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}$$

Under some certain conditions, the inverse  $B^{r,s} = B^{r,s-1} = (b_{nk}^{r,s})^{-1}$  of the matrix  $B^{r,s} = (b_{nk}^{r,s})$  is regular.

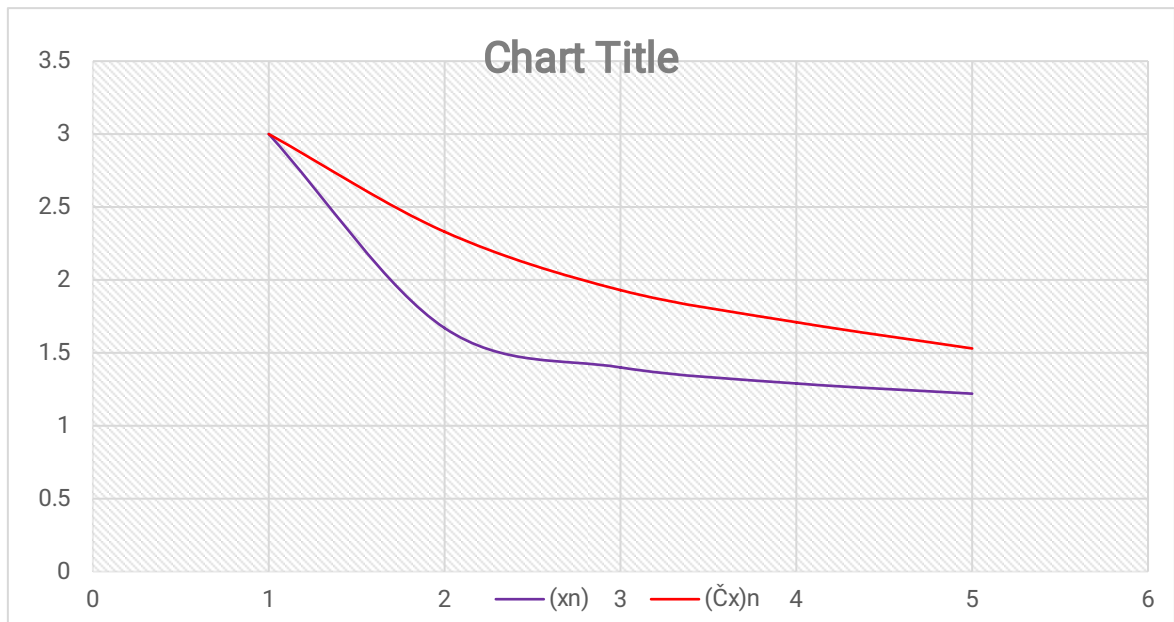
**EXAMPLE B<sub>1</sub>:** Let  $x_n = \frac{2n+1}{2n-1} \forall n \in \mathbb{N}, \beta < n$  be a sequence. Then  $\lim_{n \rightarrow \infty} x_n \rightarrow 1$

Further,

$$(B^{s,s}x) = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & \dots \\ \frac{1}{2} & \frac{1}{2} & 0 & 0 & 0 & 0 & \dots \\ \frac{1}{4} & \frac{1}{2} & \frac{1}{4} & 0 & 0 & 0 & \dots \\ \frac{1}{8} & \frac{3}{8} & \frac{3}{8} & \frac{1}{8} & 0 & 0 & \dots \\ \frac{1}{16} & \frac{1}{4} & \frac{3}{8} & \frac{1}{4} & \frac{1}{16} & 0 & \dots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix} \begin{pmatrix} 3 \\ \frac{5}{3} \\ 7 \\ 5 \\ 9 \\ 7 \\ 11 \\ 9 \\ \vdots \end{pmatrix} = \begin{pmatrix} 3+0+\dots \\ \frac{3}{2}+\frac{5}{6}+0+\dots \\ \frac{3}{4}+\frac{5}{6}+\frac{7}{20}+0+\dots \\ \frac{3}{8}+\frac{5}{8}+\frac{21}{40}+\frac{3}{16}+0+\dots \\ \frac{3}{16}+\frac{5}{12}+\frac{21}{40}+\frac{9}{28}+\frac{11}{144}+0+\dots \\ \vdots \end{pmatrix} = \begin{pmatrix} 3 \\ \frac{14}{6} \\ \frac{29}{15} \\ \frac{137}{80} \\ \frac{481}{315} \\ \vdots \end{pmatrix}$$

$$\Rightarrow (B^{r,s}x) = \begin{pmatrix} 3 \\ \frac{14}{6} \\ \frac{29}{15} \\ \frac{137}{80} \\ \frac{481}{315} \\ \vdots \end{pmatrix} = \begin{pmatrix} 3 \\ 2+\frac{1}{3} \\ 1+\frac{14}{15} \\ 1+\frac{57}{80} \\ 1+\frac{166}{315} \\ \vdots \end{pmatrix}$$

Clearly,  $\lim_{n \rightarrow \infty} (B^{r,s}x)_n \rightarrow 1$

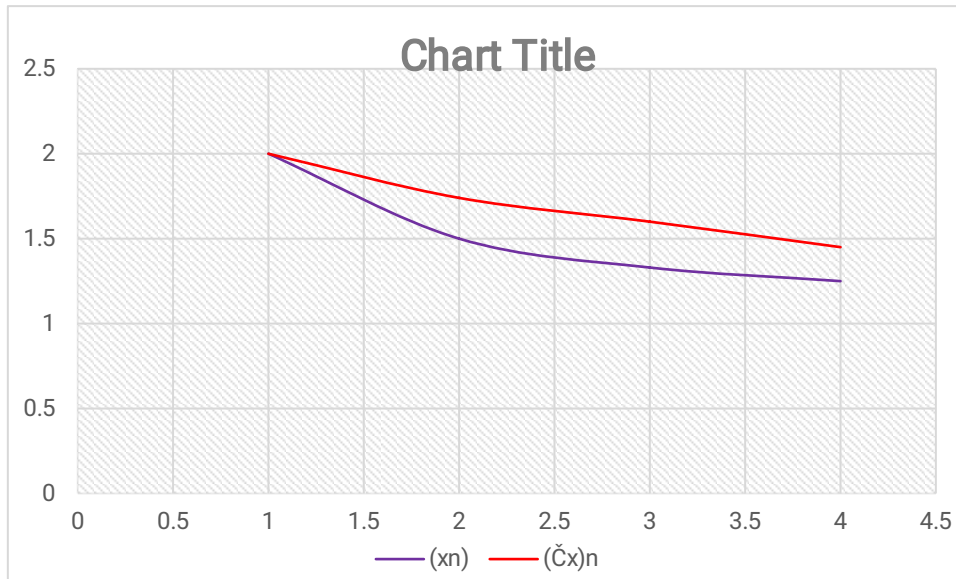


**Example B2:** Let  $x_n = \frac{n+1}{n} \forall n \in \mathbb{N}, \beta < n$  be a sequence. Then  $\lim_{n \rightarrow \infty} x_n = 1$

Further,

$$\begin{aligned}
 (B^{\frac{3}{4}, \frac{2}{3}} x)_n &= \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & \dots \\ \frac{8}{17} & \frac{9}{17} & 0 & 0 & 0 & 0 & \dots \\ \frac{64}{289} & \frac{144}{289} & \frac{81}{289} & 0 & 0 & 0 & \dots \\ \frac{512}{4913} & \frac{1728}{4913} & \frac{1944}{4913} & \frac{729}{4913} & 0 & 0 & \dots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix} \begin{pmatrix} 2 \\ 3 \\ 2 \\ 4 \\ 3 \\ 5 \\ 4 \\ \vdots \end{pmatrix} \\
 &= \begin{pmatrix} 2+0+\dots \\ \frac{16}{17} + \frac{27}{34} + 0 + \dots \\ \frac{128}{289} + \frac{216}{289} + \frac{324}{867} + 0 + \dots \\ \frac{1024}{4913} + \frac{5184}{9826} + \frac{7776}{14739} + \frac{3645}{19652} + 0 + \dots \\ \vdots \end{pmatrix} = \begin{pmatrix} 2 \\ \frac{59}{34} \\ \frac{1386}{867} \\ \frac{85431}{58956} \\ \vdots \end{pmatrix} = \begin{pmatrix} 2 \\ 1 + \frac{25}{34} \\ 1 + \frac{519}{867} \\ 1 + \frac{8825}{19652} \\ \vdots \end{pmatrix} \\
 \Rightarrow (B^{r,s} x)_n &= \begin{pmatrix} 2 \\ 1 + \frac{25}{34} \\ 1 + \frac{519}{867} \\ 1 + \frac{8825}{19652} \\ \vdots \end{pmatrix}
 \end{aligned}$$

Clearly,  $\lim_{n \rightarrow \infty} (B^{r,s} x)_n = 1$

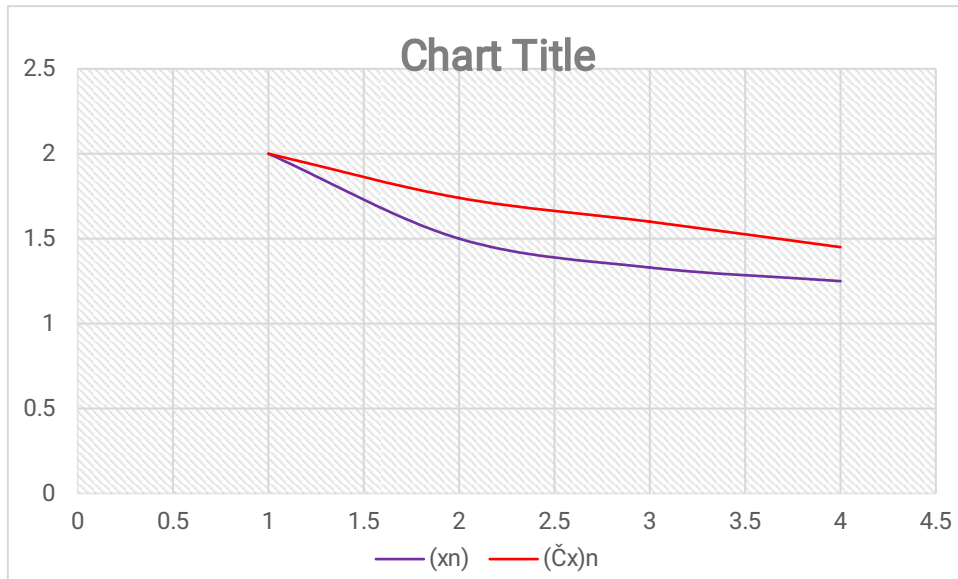


**Example B<sub>3</sub>:** Let  $x_n = \frac{n+1}{n} \forall n \in \mathbb{N}, \beta < n$  be a sequence. Then  $\lim_{n \rightarrow \infty} x_n = 1$

Further,

$$\begin{aligned}
 (B^{r,s}x) &= \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & \dots \\ \frac{15}{17} & \frac{2}{17} & 0 & 0 & 0 & 0 & \dots \\ \frac{225}{289} & \frac{60}{289} & \frac{4}{289} & 0 & 0 & 0 & \dots \\ \frac{40}{4913} & \frac{900}{4913} & \frac{1350}{4913} & \frac{3375}{4913} & 0 & 0 & \dots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \dots \end{pmatrix} \begin{pmatrix} 2 \\ 0 \\ 2 \\ 0 \\ 2 \\ \vdots \end{pmatrix} \\
 &= \begin{pmatrix} 2+0+\dots \\ \frac{30}{17}+0+0+\dots \\ \frac{450}{289}+0+\frac{8}{289}+0+\dots \\ \frac{80}{4913}+0+\frac{2700}{4913}+0+0+\dots \\ \vdots \end{pmatrix} = \begin{pmatrix} 2 \\ \frac{30}{17} \\ \frac{458}{289} \\ \frac{2780}{4913} \\ \vdots \end{pmatrix} = \begin{pmatrix} 2 \\ 1+\frac{13}{17} \\ 1+\frac{169}{289} \\ \frac{2780}{4913} \\ \vdots \end{pmatrix} = \begin{pmatrix} 2 \\ 1.76 \\ 1.58 \\ 0.57 \\ \vdots \end{pmatrix} \\
 \Rightarrow (B^{r,s}x) &= \begin{pmatrix} 2 \\ \frac{30}{17} \\ \frac{458}{289} \\ \frac{2780}{4913} \\ \vdots \end{pmatrix}
 \end{aligned}$$

Clearly,  $\lim_{n \rightarrow \infty} (B^{r,s}x)_n = 1$



**Lemma 4.2:** Let  $r, s$  be two non zero real numbers with  $r + s \neq 0$  and let  $B^{r,s} = (b_{nk}^{r,s})$  be the binomial matrix defined above. Then,  $B^{r,s} = (b_{nk}^{r,s})$  is regular if and only if  $r, s \neq 0$  and  $r$  and  $s$  are non-conjugate.

**Proof:**

Suppose on the contrary that either  $r = 0$  and  $s \neq 0$  or both  $r = 0$  and  $s = 0$ . Then, it will follow from the fact that  $r + s \neq 0$  and **vii** that

$$b_{nk}^{0,0} = \begin{cases} \frac{1}{(0)^n} \binom{n}{k} 0^{n-k} 0^k, & 0 \leq k \leq n \\ 0, & k > n \end{cases} = \begin{cases} \infty \cdot \binom{n}{k} \cdot 0 \cdot 0, & 0 \leq k \leq n \\ 0, & k > n \end{cases} = \begin{cases} \infty, & 0 \leq k \leq n \\ 0, & k > n \end{cases} \quad \forall k, n \in \mathbb{N}^0$$

$$\Rightarrow b_{nk}^{0,0} = \infty$$

And if one of them is zero, then, it will also follow that

$$b_{nk}^{r,0} = \begin{cases} \frac{1}{(r)^n} \binom{n}{k} 0^{n-k} r^k, & 0 \leq k \leq n \\ 0, & k > n \end{cases} = \begin{cases} 0, & 0 \leq k \leq n \\ 0, & k > n \end{cases} \quad \forall k, n \in \mathbb{N}^0$$

Thus, this contradict the fact that

$$b_{nk}^{r,s} = \begin{cases} \frac{1}{(s+r)^n} \binom{n}{k} s^{n-k} r^k, & 0 \leq k \leq n \\ 0, & k > n \end{cases}$$

$\forall k, n \in \mathbb{N}$  and  $r, s$  are non zero real numbers with  $r + s \neq 0$ . Thus, Toeplitz conditions would all fail and hence,  $r$  and  $s$  cannot be zero (0)

Now therefore, suppose that  $r, s \neq 0 \in \mathbb{R}$ . Then, it will follow from **i-v, viii**, and from the fact that  $r$  and  $s$  have the same sign that

$$\text{i. } \sum_{k=1}^{\infty} |\check{c}_{nk}| = \sum_{k=1}^{\infty} \left| \frac{1}{(s+r)^n} \binom{n}{k} s^{n-k} r^k \right| = \left| \frac{(n)! s^n}{(s+r)^n} \sum_{k=1}^{\infty} \frac{1}{(n-k)!} \left( \frac{r}{s} \right)^k \right| = \left| \frac{(n)! s^n}{(s+r)^n k!} \cdot R_m \right| = M$$

$\Rightarrow \sum_{k=1}^{\infty} |\check{c}_{nk}| = M < \infty$  for each  $n = 1, 2, \dots$  fulfilling condition (i) of Toeplitz matrix.

$$\text{ii. } \lim_{n \rightarrow \infty} b_{nk}^{r,s} = \lim_{n \rightarrow \infty} \frac{1}{(s+r)^n k!} \binom{n}{k} s^{n-k} r^k = \frac{1}{k!} \left( \frac{r}{s} \right)^k \lim_{n \rightarrow \infty} \frac{(n)! s^n}{(s+r)^n (n-k)}$$

$$= R_k \lim_{n \rightarrow \infty} \frac{(n)! s^n}{(s+r)^n (n-k)} = R_k \lim_{n \rightarrow \infty} \frac{1}{(s+r)^n (n-k)} \cdot (n)! s^n = R_k \cdot 0 = 0$$

$\Rightarrow \lim_{n \rightarrow \infty} b_{nk}^{r,s} = 0$  (This is condition (ii) Toeplitz matrix).

$$\text{iii. } \lim_{n \rightarrow \infty} \sum_{k=1}^{\infty} b_{nk}^{r,s} = \lim_{n \rightarrow \infty} \sum_{k=1}^{\infty} \frac{1}{(s+r)^n k!} \binom{n}{k} s^{n-k} r^k = \lim_{n \rightarrow \infty} \frac{(n)! s^n}{(s+r)^n} \sum_{k=1}^{\infty} \frac{1}{k! (n-k)!} \left( \frac{r}{s} \right)^k = \lim_{n \rightarrow \infty} \frac{(n)! s^n}{(s+r)^n} \cdot R^n$$

$$\text{Where } R^n = \sum_{k=1}^{\infty} \frac{1}{k! (n-k)!} \left( \frac{r}{s} \right)^k$$

$$\Rightarrow \lim_{n \rightarrow \infty} \sum_{k=1}^{\infty} b_{nk}^{r,s} \lim_{n \rightarrow \infty} \frac{(n)! s^n R^n}{(s+r)^n} \neq 1$$

Now, if  $r, s > -1$ , then,

$$\text{a. } \lim_{n \rightarrow \infty} \sum_{k=1}^{\infty} b_{nk}^{r,s} = \lim_{n \rightarrow \infty} \sum_{k=1}^{\infty} \frac{1}{(s+r)^n k!} \binom{n}{k} s^{n-k} r^k = \lim_{n \rightarrow \infty} \frac{(n)! s^n}{(s+r)^n} \sum_{k=1}^{\infty} \frac{1}{k! (n-k)!} \left( \frac{r}{s} \right)^k = \lim_{n \rightarrow \infty} \frac{(n)! s^n}{(s+r)^n} R^n$$

$$\text{Where } R^n = \sum_{k=1}^{\infty} \frac{1}{k! (n-k)!} \left( \frac{r}{s} \right)^k = \frac{(n)! s^n}{(s+r)^n} \text{ by i-v and viii.}$$

$$\Rightarrow \lim_{n \rightarrow \infty} \sum_{k=1}^{\infty} b_{nk}^{r,s} = \lim_{n \rightarrow \infty} 1 = 1$$

$$\Rightarrow \lim_{n \rightarrow \infty} \sum_{k=1}^{\infty} b_{nk}^{r,s} = 1 \text{ (This is nothing but condition iii Toeplitz matrix)}$$

Also, if  $r = s$ , then, it will follow automatically from i above that

$\lim_{n \rightarrow \infty} \sum_{k=1}^{\infty} b_{nk}^{r,s} = \lim_{n \rightarrow \infty} 1 = 1$  (This satisfies condition iii Toeplitz matrix)

**Theorem 4.4:** Let  $r$  and  $s$  be any two real numbers with the property that  $r + s \neq 0$  and let

$C_n = \frac{(2n)!}{n!(n+1)!} \forall n, k \in \mathbb{N}^0$  be a Catalan number. Then, the matrix  $A(\check{C}, B^{r,s}) = (\check{c}_{nk}^{r,s})$  where

$$\check{c}_{nk}^{r,s} = \begin{cases} \frac{c_k c_{n-k}}{c_{n+1}} \binom{n}{k} s^{n-k} r^k, & 0 \leq k \leq n \\ 0, & k > n \end{cases} \quad \text{is regular if and only if } r, s \neq 0 .$$

**Proof**

$$(\check{C}, B^{r,s}) A(\check{C}, B^{r,s})^{-1} = I$$



$$\Rightarrow \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & \dots \\ \frac{s}{c_2(r+s)} & \frac{r}{c_2(r+s)} & 0 & 0 & 0 & \dots \\ \frac{c_2 s^2}{c_3(r+s)^2} & \frac{c_2 sr}{c_3(r+s)^2} & \frac{c_2 r^2}{c_3(r+s)^2} & 0 & 0 & \dots \\ \frac{c_3 s^3}{c_4(r+s)^3} & \frac{(c_3+c_1)s^2 r}{c_4(r+s)^3} & \frac{(c_3+c_1)sr^2}{c_4(r+s)^3} & \frac{c_3 r^3}{c_4(r+s)^3} & 0 & \dots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix} A(\check{C}, B^{r,s})^{-1} = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & \dots \\ 0 & 1 & 0 & 0 & 0 & 0 & \dots \\ 0 & 0 & 1 & 0 & 0 & 0 & \dots \\ 0 & 0 & 0 & 1 & 0 & 0 & \dots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}$$

$$\Rightarrow \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & \dots \\ s & r & 0 & 0 & 0 & 0 & \dots \\ c_2 s^2 & c_2 sr & c_2 r^2 & 0 & 0 & 0 & \dots \\ c_3 s^3 & (c_3+c_1)s^2 r & (c_3+c_1)sr^2 & c_3 r^3 & 0 & \dots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix} A(\check{C}, B^{r,s})^{-1} = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & \dots \\ 0 & c_2(r+s) & 0 & 0 & 0 & 0 & \dots \\ 0 & 0 & c_3(r+s)^2 & 0 & 0 & 0 & \dots \\ 0 & 0 & 0 & c_4(r+s)^3 & 0 & 0 & \dots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}$$

$$\Rightarrow \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & \dots \\ 1 & \frac{r}{s} & 0 & 0 & 0 & 0 & \dots \\ 1 & \frac{r}{s} & \frac{r^2}{s^2} & 0 & 0 & 0 & \dots \\ 1 & \frac{(c_3+c_1)r}{c_3 s} & \frac{(c_3+c_1)r^2}{c_3 s^2} & \frac{c_3 r^3}{c_3 s^3} & 0 & 0 & \dots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix} A(\check{C}, B^{r,s})^{-1} = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & \dots \\ 0 & \frac{c_2(r+s)}{s} & 0 & 0 & 0 & 0 & \dots \\ 0 & 0 & \frac{c_3(r+s)^2}{c_2 s^2} & 0 & 0 & 0 & \dots \\ 0 & 0 & 0 & \frac{c_4(r+s)^3}{c_3 s^3} & 0 & 0 & \dots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}$$

$$\Rightarrow \left( \begin{array}{cccccccc} 1 & 0 & 0 & 0 & 0 & 0 & \dots \\ 0 & \frac{r}{s} & 0 & 0 & 0 & 0 & \dots \\ 0 & 0 & \frac{r^2}{s^2} & 0 & 0 & 0 & \dots \\ 0 & \frac{r}{c_3 s} & \frac{r^2}{c_3 s^2} & \frac{r^3}{s^3} & 0 & 0 & \dots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{array} \right) A(\check{C}, B^{r,s})^{-1} = \left( \begin{array}{cccccccc} 1 & 0 & 0 & 0 & 0 & 0 & \dots \\ -1 & \frac{c_2(r+s)}{s} & 0 & 0 & 0 & 0 & \dots \\ 0 & -\frac{c_2(r+s)}{s} & \frac{c_3(r+s)^2}{c_2 s^2} & 0 & 0 & 0 & \dots \\ 0 & 0 & -\frac{c_3(r+s)^2}{c_2 s^2} & \frac{c_4(r+s)^3}{c_3 s^3} & 0 & 0 & \dots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{array} \right)$$

$$\Rightarrow \left( \begin{array}{cccccccc} 1 & 0 & 0 & 0 & 0 & 0 & \dots \\ 0 & 1 & 0 & 0 & 0 & 0 & \dots \\ 0 & 0 & 1 & 0 & 0 & 0 & \dots \\ 0 & \frac{r}{c_3 s} & \frac{r}{s} & \frac{c_3 r^2}{s^2} & 0 & 0 & \dots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{array} \right) A(\check{C}, B^{r,s})^{-1} = \left( \begin{array}{cccccccc} 1 & 0 & 0 & 0 & 0 & 0 & \dots \\ -\frac{s}{r} & \frac{c_2(r+s)}{r} & 0 & 0 & 0 & 0 & \dots \\ 0 & -\frac{c_2 s(r+s)}{r^2} & \frac{c_3 s^2(r+s)^2}{c_2 r^2} & 0 & 0 & 0 & \dots \\ 0 & 0 & -\frac{c_3^2(r+s)^2}{c_2 s} & \frac{c_4(r+s)^3}{s^2 r} & 0 & 0 & \dots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{array} \right)$$

$$\Rightarrow \left( \begin{array}{cccccccc} 1 & 0 & 0 & 0 & 0 & 0 & \dots \\ 0 & 1 & 0 & 0 & 0 & 0 & \dots \\ 0 & 0 & 1 & 0 & 0 & 0 & \dots \\ 0 & 1 & \frac{r}{s} & \frac{c_3 r^2}{s^2} & 0 & 0 & \dots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{array} \right) A(\check{C}, B^{r,s})^{-1} = \left( \begin{array}{cccccccc} 1 & 0 & 0 & 0 & 0 & 0 & \dots \\ -\frac{s}{r} & \frac{c_2(r+s)}{r} & 0 & 0 & 0 & 0 & \dots \\ 0 & -\frac{c_2 s(r+s)}{r^2} & \frac{c_3 s^2(r+s)^2}{c_2 r^2} & 0 & 0 & 0 & \dots \\ 0 & 0 & -\frac{c_3^2(r+s)^2}{c_2 s} & \frac{c_4(r+s)^3}{s^2 r} & 0 & 0 & \dots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{array} \right)$$

$$\Rightarrow \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & \dots \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & \dots \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & \dots \\ 0 & 0 & \frac{r}{s} & \frac{c_3 r^2}{s^2} & 0 & 0 & \dots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix} A(\check{C}, B^{r,s})^{-1} = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & \dots \\ -\frac{s}{r} & \frac{c_2(r+s)}{r} & 0 & 0 & 0 & 0 & \dots \\ 0 & \frac{-c_2 s(r+s)}{r^2} & \frac{c_3 s^2 (r+s)^2}{c_2 r^2} & 0 & 0 & 0 & \dots \\ 0 & \frac{s}{r} & \frac{-(r+s)(c_3^2 r(r+s) - c_2)}{c_2 s r} & \frac{c_4 (r+s)^3}{s^2 r} & 0 & \dots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}$$

$$\Rightarrow \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & \dots \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & \dots \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & \dots \\ 0 & 0 & 1 & \frac{c_3 r}{s} & 0 & 0 & \dots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix} A(\check{C}, B^{r,s})^{-1} = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & \dots \\ -\frac{s}{r} & \frac{c_2(r+s)}{r} & 0 & 0 & 0 & 0 & \dots \\ 0 & \frac{-c_2 s(r+s)}{r^2} & \frac{c_3 s^2 (r+s)^2}{c_2 r^2} & 0 & 0 & 0 & \dots \\ 0 & \left(\frac{s}{r}\right)^2 & \frac{-(r+s)(c_3^2 r(r+s) - c_2)}{c_2 r^2} & \frac{c_4 (r+s)^3}{s r^2} & 0 & \dots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}$$

$$\Rightarrow \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & \dots \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & \dots \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & \dots \\ 0 & 0 & 0 & \frac{c_3 r}{s} & 0 & 0 & \dots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix} A(\check{C}, B^{r,s})^{-1} = \begin{pmatrix} 1 & 0 & 0 & 0 & \dots \\ -\frac{s}{r} & \frac{c_2(r+s)}{r} & 0 & 0 & \dots \\ 0 & \frac{-c_2 s(r+s)}{r^2} & \frac{c_3 s^2 (r+s)^2}{c_2 r^2} & 0 & \dots \\ 0 & \frac{s^2 - c_2 s(r+s)}{r^2} & \frac{-(r+s)[(c_3^2 r(r+s) - c_2) - c_3 s^2 (r+s)]}{c_2 r^2} & \frac{c_4 (r+s)^3}{s r^2} & \dots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}$$

$$\begin{pmatrix} 1 & 0 & 0 & 0 & 0 & \dots & \dots \\ 0 & 1 & 0 & 0 & 0 & \dots & \dots \\ 0 & 0 & 1 & 0 & 0 & \dots & \dots \\ 0 & 0 & 0 & 1 & 0 & \dots & \dots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \ddots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \ddots \end{pmatrix} A(\check{C}, B^{r,s})^{-1} = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & \dots \\ \frac{s}{r} & \frac{-c_2(r+s)}{r} & 0 & 0 & 0 & 0 & \dots \\ 0 & \frac{-c_2 s(r+s)}{r^2} & \frac{c_3 s^2 (r+s)^2}{c_2 r^2} & 0 & 0 & 0 & \dots \\ 0 & \frac{s^2 (s - c_2 (r+s))}{c_3 r^3} & \frac{-s(r+s)[(c_3^2 r(r+s) - c_2) - c_3 s^2 (r+s)]}{c_2 c_3 r^3} & \frac{c_4 (r+s)^3}{c_3 r^3} & \dots & \dots & \dots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}$$

Thus, the Catalan Binomial matrix given by

$$A(\check{C}, B^{r,s}) = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & \dots \\ \frac{s}{c_2(r+s)} & \frac{r}{c_2(r+s)} & 0 & 0 & 0 & \dots \\ \frac{c_2 s^2}{c_3(r+s)^2} & \frac{c_2 s r}{c_3(r+s)^2} & \frac{c_2 r^2}{c_3(r+s)^2} & 0 & 0 & \dots \\ \frac{c_3 s^3}{c_4(r+s)^3} & \frac{(c_3 + c_1) s^2 r}{c_4(r+s)^3} & \frac{(c_3 + c_1) s r^2}{c_4(r+s)^3} & \frac{c_3 r^3}{c_4(r+s)^3} & 0 & \dots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}$$



**Theorem 4.5:** The matrix  $A(\check{C}, B^{r,s}) = (\check{c}_{nk}^{b^{r,s}})$  defined in theorem 4.1 above where

$$\check{c}_{nk}^{b^{r,s}} = \begin{cases} \frac{c_k c_{n-k}}{c_{n+1}} \binom{n}{k} s^{n-k} r^k, & 0 \leq k \leq n \\ 0, & k > n \end{cases} \quad \text{is regular if and only if the series } \sum_k \frac{c_k c_{n-k} r^k}{k!(n-k)!s^k} \text{ is summable to } P_n \text{ where}$$

$$P_n = \frac{c_{n+1} (s+r)^n}{n! s^n}.$$

**Proof:**

Supposed that  $A(\check{C}, B^{r,s}) = (\check{c}_{nk}^{b^{r,s}})$  is regular. Then, it will follow that:

$$\text{i. } \sup \sum_{k=1}^{\infty} |\check{c}_{nk}^{b^{r,s}}| = \sup \sum_k \left| \frac{c_k c_{n-k}}{c_{n+1}} \binom{n}{k} s^{n-k} r^k \right| = \sup \frac{n! s^n}{c_{n+1} (s+r)^n} \sum_k \left| \frac{c_k c_{n-k} r^k}{k! s^k} \right|$$

$$\Rightarrow \sup \sum_{k=1}^{\infty} |\check{c}_{nk}^{b^{r,s}}| = \sup \frac{n! s^n}{c_{n+1} (s+r)^n} \sum_k \left| \frac{c_k c_{n-k} r^k}{k!(n-k)!s^k} \right| = \sup \frac{n! s^n P_n}{c_{n+1} (s+r)^n} = M < \infty$$

$$\text{ii. } \lim_{n \rightarrow \infty} \check{c}_{nk}^{b^{r,s}} = \lim_{n \rightarrow \infty} \frac{c_k c_{n-k}}{c_{n+1}} \binom{n}{k} s^{n-k} r^k = \frac{c_k r^k}{k! s^k} \lim_{n \rightarrow \infty} \frac{n! c_{n-k} s^n}{(n-k)! c_{n+1} (s+r)^n} = \frac{c_k r^k}{k! s^k} \cdot 0 = 0$$

$$\text{iii. } \lim_{n \rightarrow \infty} \sum_k \check{c}_{nk}^{b^{r,s}} = \lim_{n \rightarrow \infty} \sum_k \frac{c_k c_{n-k}}{c_{n+1}} \binom{n}{k} s^{n-k} r^k = \lim_{n \rightarrow \infty} \frac{n! s^n}{c_{n+1} (s+r)^n} \sum_k \frac{c_k c_{n-k} r^k}{k!(n-k)!s^k}$$

Now, denote  $\sum_k \frac{c_k c_{n-k} r^k}{k!(n-k)!s^k}$  by  $P_n$  such that  $P_n = \sum_k \frac{c_k c_{n-k} r^k}{k!(n-k)!s^k} = \lim_{n \rightarrow \infty} \frac{n! s^n P_n}{c_{n+1} (s+r)^n}$ .

This implies that

$$\lim_{n \rightarrow \infty} \sum_k \check{c}_{nk}^{b^{r,s}} = \lim_{n \rightarrow \infty} \frac{n! s^n P_n}{c_{n+1} (s+r)^n}$$

$$\Rightarrow \lim_{n \rightarrow \infty} \sum_k \check{c}_{nk}^{b^{r,s}} = \lim_{n \rightarrow \infty} \frac{n! s^n P_n}{c_{n+1} (s+r)^n} = 1$$

$$\Rightarrow P_n = \frac{n! s^n}{c_{n+1} (s+r)^n}$$

Conversely, suppose that the series  $\sum_k \frac{c_k c_{n-k} r^k}{k!(n-k)!s^k}$  is summable to  $P_n = \frac{c_{n+1} (s+r)^n}{n! s^n}$ . Then, it will follow that

$$\sum_k \frac{c_k c_{n-k}}{c_{n+1}} \binom{n}{k} s^{n-k} r^k = \frac{n! s^n}{c_{n+1} (s+r)^n} \sum_k \frac{c_k c_{n-k} r^k}{k!(n-k)!s^k} = \frac{n! s^n}{c_{n+1} (s+r)^n} \cdot \frac{c_{n+1} (s+r)^n}{n! s^n} = 1$$

$$\Rightarrow \lim_{n \rightarrow \infty} \sum_k \check{c}_{nk}^{b^{r,s}} = \lim_{n \rightarrow \infty} \sum_k \frac{c_k c_{n-k}}{c_{n+1}} \binom{n}{k} s^{n-k} r^k = 1 \quad \blacksquare$$

## APPLICATION

The Catalan Binomial matrix  $A(\check{C}, B^{r,s}) = (\check{c}_{nk}^{b^{r,s}})$  is roosterly displayed as follows

$$A(\check{C}, B^{r,s}) = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & \dots \\ \frac{s}{c_2(r+s)} & \frac{r}{c_2(r+s)} & 0 & 0 & 0 & 0 & \dots \\ \frac{c_2 s^2}{c_3(r+s)^2} & \frac{c_2 sr}{c_3(r+s)^2} & \frac{c_2 r^2}{c_3(r+s)^2} & 0 & 0 & 0 & \dots \\ \frac{c_3 s^3}{c_4(r+s)^3} & \frac{(c_3+c_1)s^2 r}{c_4(r+s)^3} & \frac{(c_3+c_1)sr^2}{c_4(r+s)^3} & \frac{c_3 r^3}{c_4(r+s)^3} & 0 & \dots \\ \frac{c_4 s^4}{c_5(r+s)^4} & \frac{c_3 c_3^2 s^3 r}{c_5(r+s)^4} & \frac{c_3^2 (c_3+c_1)s^2 r^2}{c_5(r+s)^4} & \frac{c_3 c_3^2 sr^3}{c_5(r+s)^4} & \frac{c_4 r^4}{c_5(r+s)^4} & \dots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}$$

Now, consider the following cases:

**Case I:** Now, if we set  $r = s$ , it will follow that

$$A(\check{C}, B^{r,s}) = (\check{c}_{nk}^{b^{r,s}}) = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & \dots \\ \frac{1}{4} & \frac{1}{4} & 0 & 0 & 0 & 0 & \dots \\ \frac{1}{10} & \frac{1}{10} & \frac{1}{10} & 0 & 0 & 0 & \dots \\ \frac{5}{112} & \frac{6}{112} & \frac{6}{112} & \frac{5}{112} & 0 & 0 & \dots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}$$

**Example 4.1:** Now, let  $x = (x_k) = (1, 0, 1, 0, 1, 0, 1, \dots)$  be a sequence.

Then, we have the function  $f$  defined by

$$f(x) = \begin{cases} 1 & \text{if } k \text{ is odd} \\ 0 & \text{if } k \text{ is even} \end{cases} \text{ for all } k \in \mathbb{N}.$$

Thus  $f$  has two fixed point, 0 and 1.

Further,  $(x_k)$  is unit porous, circlic and unit bounded. Furthermore, we know that

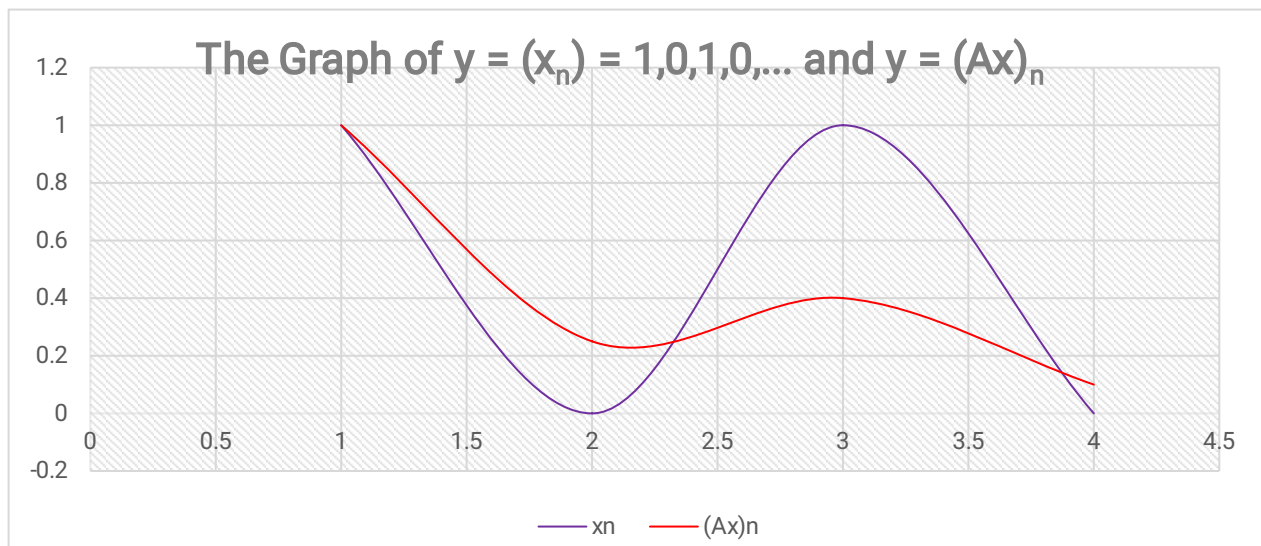
$$\lim_{k \rightarrow \infty} x_k = 0$$

Now, applying the Catalan Binomial matrix  $A(\check{C}, B^{r,s}) = (\check{c}_{nk}^{b^{r,s}})$  with the above conditions, we have

$$(A(\check{C}, B^{r,s})x)_n = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & \dots \\ \frac{1}{4} & \frac{1}{4} & 0 & 0 & 0 & \dots \\ \frac{1}{10} & \frac{1}{10} & \frac{1}{10} & 0 & 0 & \dots \\ \frac{5}{112} & \frac{6}{112} & \frac{6}{112} & \frac{5}{112} & 0 & \dots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix} \begin{pmatrix} 1 \\ 0 \\ 1 \\ 0 \\ 1 \\ 0 \\ \vdots \end{pmatrix} = \begin{pmatrix} 1+0+\dots \\ \frac{1}{4}+0+\dots \\ \frac{1}{10}+0+\frac{1}{10}+0+\dots \\ \frac{5}{112}+0+\frac{6}{112}+0 \\ \vdots \end{pmatrix} = \begin{pmatrix} 1 \\ \frac{1}{4} \\ \frac{2}{5} \\ \frac{11}{112} \\ \vdots \end{pmatrix}$$

$$\Rightarrow (A(\check{C}, B^{r,s})x)_n = \left(1, \frac{1}{4}, \frac{2}{5}, \frac{11}{112}, \dots\right)$$

Clearly,  $\lim_{n \rightarrow \infty} (\check{C}x)_n = 0$



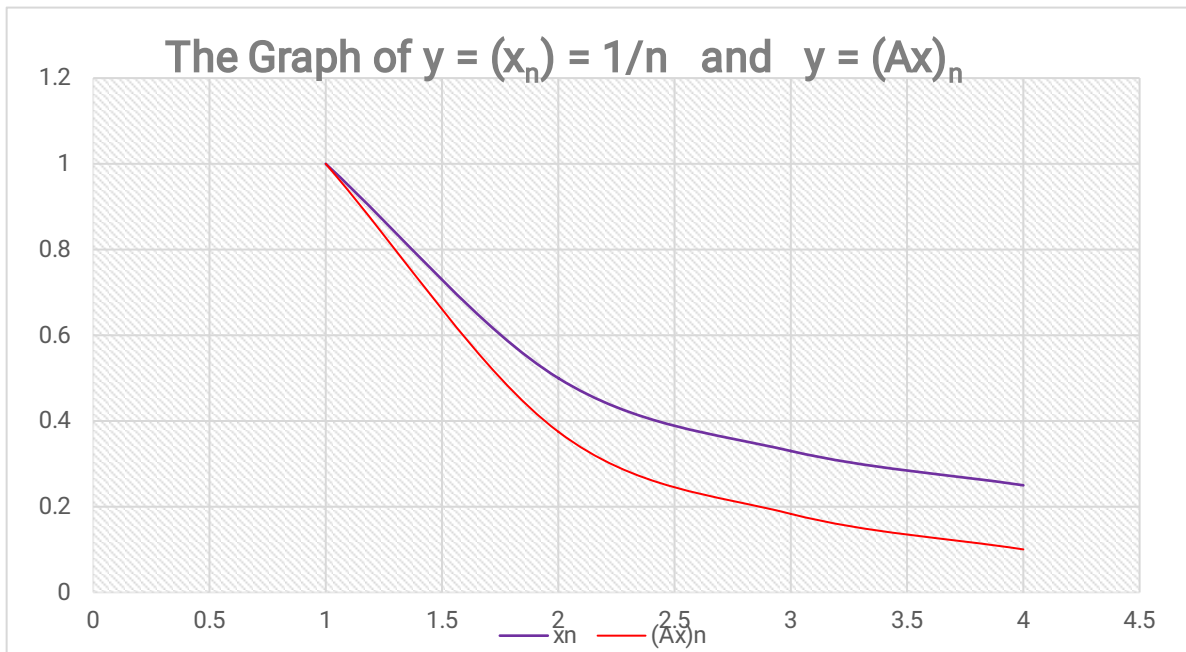
**Example 4.2:** Let  $x_k = \left(1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \dots\right) \forall n \in \mathbb{N}$ . Then,  $f(x) = \frac{1}{n}$

Thus,  $(x_k)$  is nested, non-porous and bounded above and below. Furthermore  $\lim_{k \rightarrow \infty} x_k \rightarrow 0$ .

Now, applying the Catalan Binomial matrix  $A(\check{C}, B^{r,s}) = (\check{c}_{nk}^{b^{r,s}})$  given above under the given conditions, we have

$$\begin{aligned}
 (A(\check{C}, B^{r,s})x)_n &= \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & \dots \\ \frac{1}{4} & \frac{1}{4} & 0 & 0 & 0 & 0 & \dots \\ \frac{1}{10} & \frac{1}{10} & \frac{1}{10} & 0 & 0 & 0 & \dots \\ \frac{5}{112} & \frac{6}{112} & \frac{6}{112} & \frac{5}{112} & 0 & 0 & \dots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix} \begin{pmatrix} 1 \\ \frac{1}{2} \\ \frac{1}{3} \\ \frac{1}{4} \\ \vdots \end{pmatrix} = \begin{pmatrix} 1+0+\dots \\ \frac{1}{4}+\frac{1}{8}+0+\dots \\ \frac{1}{10}+\frac{1}{20}+\frac{1}{30}+0+\dots \\ \frac{5}{112}+\frac{3}{112}+\frac{1}{56}+\frac{5}{448}+0+\dots \\ \vdots \end{pmatrix} \\
 \Rightarrow (A(\check{C}, B^{r,s})x)_n &= \begin{pmatrix} 1 \\ \frac{3}{8} \\ \frac{11}{60} \\ \frac{45}{448} \\ \vdots \end{pmatrix}
 \end{aligned}$$

Clearly,  $\lim_{n \rightarrow \infty} (A(\check{C}, B^{r,s})x)_n \rightarrow 0$



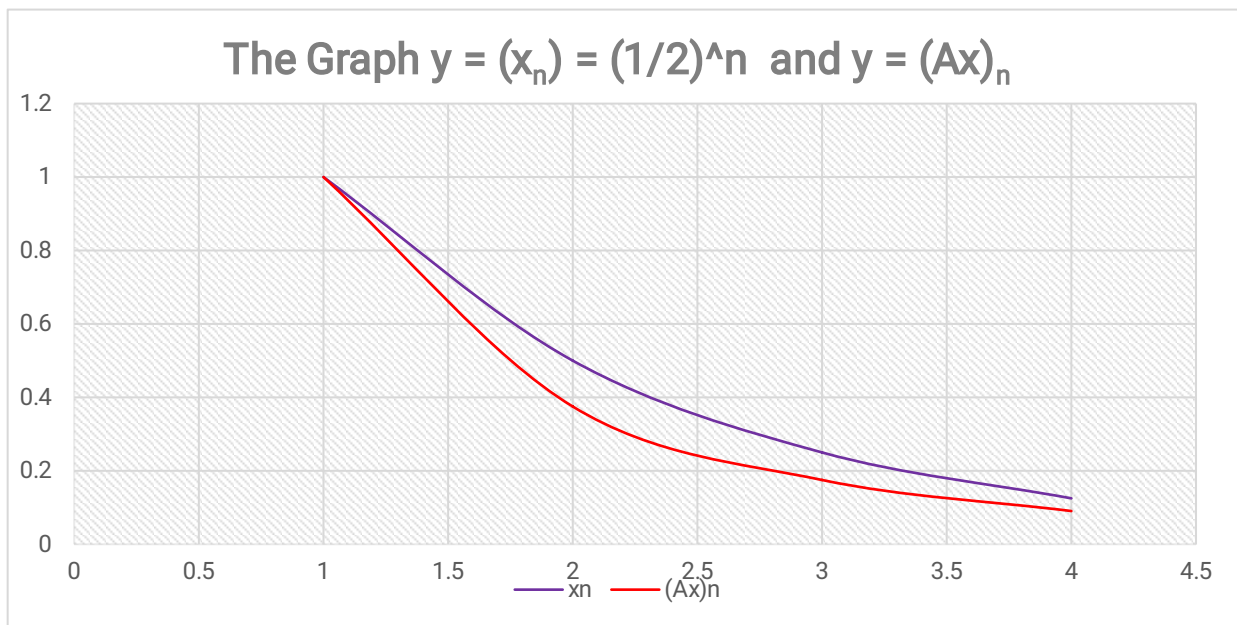
**Example 4.3:** Let  $x_n = \left(\frac{1}{2}\right)^n \forall n \in \mathbb{N}$  be a sequence. Then  $\lim_{n \rightarrow \infty} \left(\frac{1}{2}\right)^n \rightarrow 0$

Thus, applying our Catalan Binomial matrix  $A(\check{C}, B^{r,s}) = (\check{c}_{nk}^{b,r,s})$  we have

$$\begin{aligned}
 (A(\check{C}, B^{r,s})x)_n &= \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & \dots \\ \frac{1}{4} & \frac{1}{4} & 0 & 0 & 0 & 0 & \dots \\ \frac{1}{10} & \frac{1}{10} & \frac{1}{10} & 0 & 0 & 0 & \dots \\ \frac{5}{112} & \frac{6}{112} & \frac{6}{112} & \frac{5}{112} & 0 & 0 & \dots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix} \begin{pmatrix} 1 \\ \frac{1}{2} \\ \frac{1}{4} \\ \frac{1}{8} \\ \frac{1}{16} \\ \vdots \end{pmatrix} = \begin{pmatrix} 1+0+\dots \\ \frac{1}{4}+\frac{1}{8}+0+\dots \\ \frac{1}{10}+\frac{1}{20}+\frac{1}{40}+0+\dots \\ \frac{5}{112}+\frac{6}{224}+\frac{6}{448}+\frac{5}{896}+0+\dots \\ \vdots \end{pmatrix} \\
 &= \begin{pmatrix} 1 \\ \frac{3}{8} \\ \frac{7}{40} \\ \frac{81}{896} \\ \vdots \end{pmatrix} \\
 \Rightarrow (A(\check{C}, B^{r,s})x)_n &= \begin{pmatrix} 1 \\ \frac{3}{8} \\ \frac{7}{40} \\ \frac{81}{896} \\ \vdots \end{pmatrix}
 \end{aligned}$$

Clearly,  $\lim_{n \rightarrow \infty} (A(\check{C}, B^{r,s})x)_n \rightarrow 0$

**The Geometrical presentation**



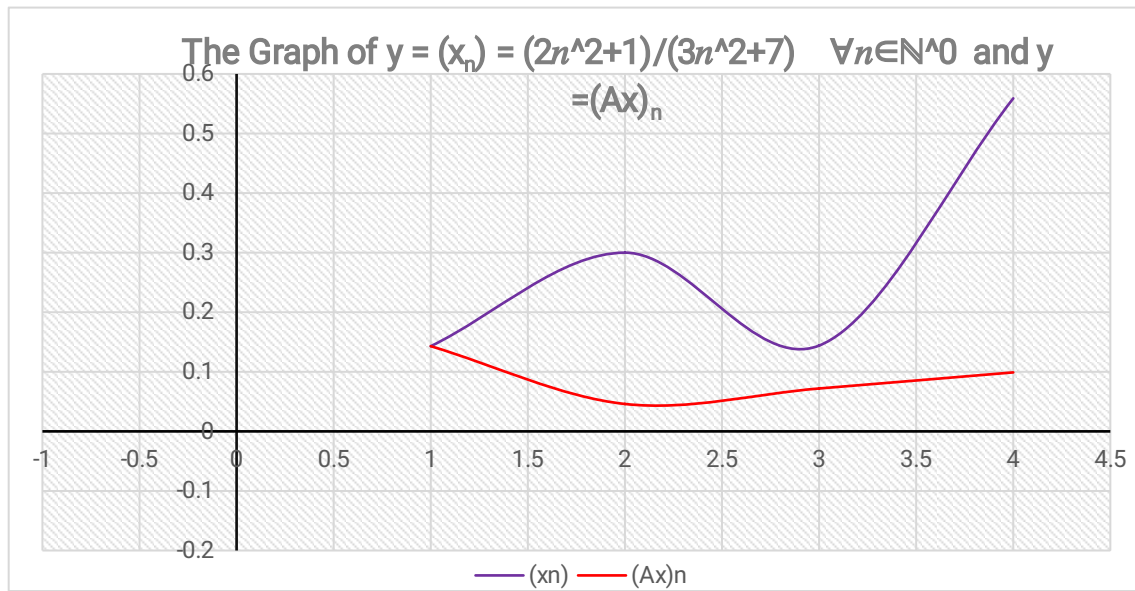
**Example 4.4:** Let  $x_n = \frac{2n^2+1}{3n^2+7} \forall n \in \mathbb{N}^0$  be a sequence. Then  $\lim_{n \rightarrow \infty} \frac{2n^2+1}{3n^2+7} \rightarrow \frac{2}{3}$ . Thus,

$$(A(\check{C}, B^{r,s})x)_n = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & \dots \\ \frac{1}{4} & \frac{1}{4} & 0 & 0 & 0 & 0 & \dots \\ \frac{1}{10} & \frac{1}{10} & \frac{1}{10} & 0 & 0 & 0 & \dots \\ \frac{5}{112} & \frac{6}{112} & \frac{6}{112} & \frac{5}{112} & 0 & 0 & \dots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix} \begin{pmatrix} \frac{1}{7} \\ \frac{3}{10} \\ \frac{9}{19} \\ \frac{19}{34} \\ \vdots \end{pmatrix} = \begin{pmatrix} \frac{1}{7} + 0 + \dots \\ \frac{1}{28} + \frac{3}{40} + 0 + \dots \\ \frac{1}{70} + \frac{3}{100} + \frac{9}{190} + 0 + \dots \\ \frac{5}{784} + \frac{18}{1120} + \frac{54}{2128} + \frac{195}{3808} + 0 + \dots \\ \vdots \end{pmatrix}$$

$$= \begin{pmatrix} \frac{1}{7} \\ \frac{13}{280} \\ \frac{953}{13300} \\ \frac{501566}{5064640} \\ \vdots \end{pmatrix}$$

$$\Rightarrow (A(\check{C}, B^{r,s})x)_n = \begin{pmatrix} \frac{1}{7} \\ \frac{13}{280} \\ \frac{953}{13300} \\ \frac{250783}{2532320} \\ \vdots \end{pmatrix}$$

Clearly,  $\lim_{n \rightarrow \infty} (A(\check{C}, B^{r,s})x)_n \rightarrow \frac{1}{7}$



**Theorem 4.6:** Let  $x = (x_k)$  be an infinite sequence and  $A(\check{C}, B^{s,s}) = (\check{c}_{nk}^{b,s,s})$  be the infinite matrix defined in theorem 4.1 above. Then, the first term of the sequence  $x = (x_k)$  is fixed under the mapping  $f: x \rightarrow (Ax)_n$  if and only if the matrix,  $A(\check{C}, B^{s,s}) = (\check{c}_{nk}^{b,s,s})$  is a Toeplitz lower triangular regular matrix.

**Proof:**

Suppose that the first term of the sequence  $(x_k)$  is the fix point of the mapping

$$f: x \rightarrow (Ax)_n = \sum_{k=1}^{\infty} \check{c}_{nk}^{b,s,s} x_k = t_n.$$

Then,  $f(x_1) = x_1$  (see examples above)

$$\Rightarrow f(x_1) = x_1 \quad (1)$$

but then,

$$f(x_1) = (Ax)_1 = \sum_{k=1}^1 \check{c}_{1,k}^{b,s,s} x_k$$

$$\Rightarrow f(x_1) = \sum_{k=1}^1 \check{c}_{1,k}^{b,s,s} x_1 \quad (2)$$

From (1) and (2) we have

$$\check{c}_{1,1}^{b,s,s} x_1 = x_1$$

$$\Rightarrow \check{c}_{1,1}^{b,s,s} = 1.$$

$$\Rightarrow \check{c}_{1,2}^{b,s,s} = 0, \check{c}_{1,3}^{b,s,s} = 0, \dots,$$

This implies that  $\sum_{k=1}^{\infty} \check{c}_{1,k}^{b,s,s} = 1$

Furthermore, for the second row, we have,  $\check{c}_{2,1}^{b,s,s} + \check{c}_{2,2}^{b,s,s} = 1$  since the first term of the sequence  $(x_k)$  is the fix point of the transformation  $f: x_k \rightarrow (Ax)_n = \sum_{k=1}^{\infty} \check{c}_{n,k}^{b,s,s} x_k = t_n$  and therefore,

$$\Rightarrow \check{c}_{2,3}^{b,s,s} = 0, \check{c}_{2,4}^{b,s,s} = 0, \dots \text{ and so on.}$$

$$\Rightarrow f(x_2) = t_2 = \check{c}_{2,1}^{b,s,s} x_1 + \check{c}_{2,2}^{b,s,s} x_2.$$

Thus inductively, we have for the  $n^{\text{th}}$  row

$$f(x_{k \leq n}) = t_n = \sum_{k=1}^n \check{c}_{n,k}^{b,s,s} x_k = \check{c}_{n,1}^{b,s,s} x_1 + \check{c}_{n,2}^{b,s,s} x_2 + \dots + \check{c}_{n,n}^{b,s,s} x_n$$

$$\Rightarrow f(x_{k \leq n}) = \check{c}_{n,1}^{b,s,s} x_1 + \check{c}_{n,2}^{b,s,s} x_2 + \dots + \check{c}_{n,n}^{b,s,s} x_n$$

$$\Rightarrow f(x_1) = \check{c}_{1,1}^{b,s,s} x_1, f(x_2) = \check{c}_{2,1}^{b,s,s} x_1 + \check{c}_{2,2}^{b,s,s} x_2, \dots, f(x_{k \leq n}) = \check{c}_{n,1}^{b,s,s} x_1 + \check{c}_{n,2}^{b,s,s} x_2 + \dots + \check{c}_{n,n}^{b,s,s} x_n$$

$$\Rightarrow \check{c}_{1,1}^{b,s,s} = 1, \check{c}_{2,1}^{b,s,s} + \check{c}_{2,2}^{b,s,s} = 1, \dots, \check{c}_{n,1}^{b,s,s} + \check{c}_{n,2}^{b,s,s} + \dots + \check{c}_{n,n}^{b,s,s} = 1, \dots$$

This implies that,  $A(\check{C}, B^{s,s}) = (\check{c}_{nk}^{b,s,s})$  is a lower triangular regular matrix. But then,  $\sum_{k=1}^{\infty} \check{c}_{nk}^{b,s,s} = 1$  is a Toeplitz condition.

Hence,  $A(\check{C}, B^{s,s}) = (\check{c}_{nk}^{b,s,s})$  is Toeplitz lower triangular matrix.

Conversely, suppose that,  $A(\check{C}, B^{s,s}) = (\check{c}_{nk}^{b,s,s})$  is an infinite Toeplitz matrix and  $(x_k)$  be an infinite sequence such that  $f: x_k \rightarrow (Ax)_n = \sum_{k=1}^{\infty} \check{c}_{nk}^{b,s,s} x_k$  is a mapping from the sequence  $(x_k)$  to the sequence  $(t_n)$ . Then,  $\sum_{k=1}^{\infty} \check{c}_{nk}^{b,s,s} = 1$  since  $A(\check{C}, B^{s,s}) = (\check{c}_{nk}^{b,s,s})$  is a Toeplitz matrix but then,  $A(\check{C}, B^{s,s}) = (\check{c}_{nk}^{b,s,s})$  is a lower triangular matrix which implies that

$$\sum_{k=1}^n \check{c}_{nk}^{b,s} = 1 \text{ for } k \leq n$$

This implies that  $\check{c}_{1,1}^{b,s} = 1$  and

$$\Rightarrow \check{c}_{1,2}^{b,s} = 0, \check{c}_{1,3}^{b,s} = 0, \dots,$$

$$\Rightarrow f(x_1) = t_1 = \check{c}_{1,1}^{b,s} x_1$$

Furthermore, for the second row, we have,  $\check{c}_{2,1}^{b,s} + \check{c}_{2,2}^{b,s} = 1$

$$\Rightarrow \check{c}_{2,3}^{b,s} = 0, \check{c}_{2,4}^{b,s} = 0, \dots \text{ and so on.}$$

$$\Rightarrow f(x_2) = t_2 = \check{c}_{2,1}^{b,s} x_1 + \check{c}_{2,2}^{b,s} x_2.$$

Thus inductively, we have for the  $n^{\text{th}}$  row

$$f(x_{k \leq n}) = t_n = \sum_{k=1}^n \check{c}_{2,k}^{b,s} x_k = a_{21} x_1 + \check{c}_{2,2}^{b,s} x_2 + \dots + \check{c}_{2,2}^{b,s} x_n$$

$$\Rightarrow f(x_{k \leq n}) = \check{c}_{2,2}^{b,s} x_1 + \check{c}_{2,2}^{b,s} x_2 + \dots + \check{c}_{n,n}^{b,s} x_n$$

$$\Rightarrow \check{c}_{n,1}^{b,s} + \check{c}_{n,2}^{b,s} + \dots + \check{c}_{n,n}^{b,s} = 1$$

$$\Rightarrow f(x_{k \leq n}) = t_n = \sum_{k=1}^n \check{c}_{2,k}^{b,s} x_k = \check{c}_{2,1}^{b,s} x_1 + \check{c}_{2,2}^{b,s} x_2 + \dots + \check{c}_{n,n}^{b,s} x_n$$

Thus we have,  $t_1, t_2, t_3, \dots, t_n, \dots$  corresponding to  $x_1, x_2, x_3, \dots, x_k, \dots$ ,

But then,

$$f(x_1) = t_1 = \check{c}_{1,1}^{b,s} x_1$$

$$\Rightarrow f(x_1) = x_1$$

$$\Rightarrow f(x_1) = t_1 = x_1$$

$$\Rightarrow t_1, t_2, t_3, \dots, t_n, \dots \text{ corresponding to } x_1, x_2, x_3, \dots, x_k, \dots$$

Which clearly shows that  $f(x_1) = x_1$  to imply that  $x_1$  is the fix point of the transformation

$$f(x_k) = t_n = \sum_{k=1}^{\infty} \check{c}_{nk}^{b,s} x_k \text{ and this proves the result.}$$

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