ON A VARIANT OF BROCARD'S PROBLEM VIA THE DIAGONALIZATION METHOD

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Abstract. In this paper we introduce and develop the method of diagonalization of functions $f : \mathbb{N} \longrightarrow \mathbb{R}$. We apply this method to show that the equations of the form $\Gamma_r(n) + k = m^2$ has a finite number of solutions $n \in \mathbb{N}$ with $n > r$ for a fixed $k, r \in \mathbb{N}$, where $\Gamma_r(n) = n(n-1)\cdots(n-r)$ is the r^{th} truncated Gamma function.

1. **Introduction and problem statement**

Brocard's problem is an unsolved puzzle that questions whether the collection of integers whose factorials are the unit left translation of a square is finite or infinite. The issue was first defined by Henri Brocard, a French mathematician, between 1876 and 1885, and then rediscovered by Srinivasa Ramanujan, an Indian mathematician, in 1913. The issue is stated in a more formal way.

Problem 1.1. Does the equation $n!+1 = m^2$ has integer solutions other than $4, 5, 7$?

Other solutions to Brocard's equation are usually assumed to be finite if they exist. Indeed, Paul Erdős hypothesized that Brocard's equation had no further solutions. It has also been computationally confirmed for numbers up to 10^9 as a possible solution to the problem [4]. Although the problem has not been solved, significant theoretical progress has been achieved. M. Overholt achieved the first important breakthrough by demonstrating that there are only a finite number of integer solutions to the Brocard problem $n!+1 = m^2$ by assuming the ABC conjecture [1]. These results have also been extended to equations of the kind $n!+A = m^2$ [2]. By assuming the ABC conjecture, it is proven that the equation $n! = P(x)$, where $P(x)$ is a polynomial of degree at least two, also has a finite number of solutions. A further extension has been made in [3] where it is shown that the equation $n! = P(x)$, where $P(x)$ is a polynomial of degree at least two, also has a finite number of solution by assuming the ABC conjecture.

In this paper we apply the method of **diagonalization** of functions to show that the equation $\Gamma_r(n) + k = n^2$ has a finite number of solutions for $n \in \mathbb{N}$.

1.1. **Notations.** In this paper, we will write $f(n) \ll g(n)$ to mean there exists an absolute constant $c > 0$ such that for all sufficiently large n, then $f(n) \leq c|g(n)|$. Conversely, we will write $f(n) \gg g(n)$ if the reverse inequality holds for all sufficiently large values of n. If both inequalities hold then we write in simple terms $f(n) \asymp g(n)$.

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2. **The notion of diagonalization**

In this section we introduce and study the notion of **diagonalization** of a function. We study this notion together with associated statistics and explore some applications.

Definition 2.1. Let $f : \mathbb{N} \longrightarrow \mathbb{R}$. Then we say f is k - step **diagonalizable** at the spot $n \in \mathbb{N}$ if there exists some $m \in \mathbb{N}$ such that

$$f(n) + k = m^2.$$

We call the set of all spots $n \in \mathbb{N}$ such that f is k - step diagonalizable the k^{th} - step diagonal of f and denote by $\mathcal{D}_k(f)$. We call the set of all truncated spots $\mathcal{D}_k(f) \cap \mathbb{N}_s := \mathcal{D}_k(f, s)$ the s^{th} scale diagonal. We call the set of all squares

$$\mathbb{B}_k(f) := \left\{m^2 \in \mathbb{N} \mid f(n) + k = m^2\right\}$$

the k^{th}-step diagonal squares. We write the length of this diagonal as

$$|\mathcal{D}_k(f, s)| := \#\{n \leq s \mid f(n) + k = m^2\}.$$

It is easy to see that $|\mathcal{D}_k(f, s)| < s$.

2.1. **The s-level trace of the diagonal.**

In this section we introduce the notion of the trace of the diagonal. We launch and examine the following languages.

Definition 2.2. By the s^{th} level trace of the diagonal $\mathcal{D}_k(f)$, denoted $\mathbb{T}_f(s, k)$, we mean the partial sum

$$\mathbb{T}_f(s, k) := \sum_{\substack{n \leq s \\ n \in \mathcal{D}_k(f)}} f(n).$$

Let us suppose that f is a function with continuous derivative on $[1, s]$ for $s \geq 1$ with $s \in \mathbb{R}$, then by applying the Stieltjes integration by parts, we can write the s^{th} level trace of the diagonal in the form

$$\begin{aligned}\mathbb{T}_f(s, k) :&= \sum_{\substack{n \leq s \\ n \in \mathcal{D}_k(f)}} f(n) \\ &= \int_{1^-}^{s} f(t) d|\mathcal{D}_k(f, t)| \\ &= f(s)|\mathcal{D}_k(f, s)| - \int_{1}^{s} f'(t)|\mathcal{D}_k(f, t)| dt.\end{aligned}$$

Theorem 2.3 (Diagonal inequality)**.** *Let f be a function with continuous derivative on $[1, s]$ for $s \geq 1$ with $s \in \mathbb{R}$. If*

$$\mathcal{D}_k(f, s) - \frac{1}{f(s)}\left(\sqrt{\int_{1}^{s} |f'(t)|^2 dt}\right) \times \left(\int_{1}^{s} |\mathcal{D}_k(f, t)|^2 dt\right)^{\frac{1}{2}} \geq 0$$

for all $s \geq 1$ then the inequality holds

$$(\int\limits_{1}^{s}|\mathcal{D}_k(f,t)|^2dt)^{\frac{1}{2}} \ll \bigg(\frac{1}{f(s)}\sum_{n\leq s}f(n)\bigg)\bigg(1-\frac{1}{f(s)}\sqrt{\int\limits_{1}^{s}|f'(t)|^2dt}\bigg)^{-1}.$$

Proof. By appealing to the ensuing discussion, we obtain the upper bound

$$|\mathcal{D}_k(f,s)| \leq \frac{1}{f(s)}\sum_{n\leq s}f(n)+\frac{1}{f(s)}\int\limits_{1}^{s}f'(t)|\mathcal{D}_k(f,t)|dt$$

so that by appealing to the Cauchy-Schwartz inequality, we obtain further the upper bound

$$|\mathcal{D}_k(f,s)| \leq \frac{1}{f(s)}\sum_{n\leq s}f(n)+\frac{1}{f(s)}\bigg(\int\limits_{1}^{s}|f'(t)|^2dt\bigg)^{\frac{1}{2}}\times\bigg(\int\limits_{1}^{s}|\mathcal{D}_k(f,t)|^2dt\bigg)^{\frac{1}{2}}.$$

By rearranging terms, appealing to the condition

$$\mathcal{D}_k(f,s)-\frac{1}{f(s)}\bigg(\sqrt{\int\limits_{1}^{s}|f'(t)|^2dt}\bigg)\times\bigg(\int\limits_{1}^{s}|\mathcal{D}_k(f,t)|^2dt\bigg)^{\frac{1}{2}}\geq 0$$

and noting that

$$(\int\limits_{1}^{s}|\mathcal{D}_k(f,t)|^2dt)^{\frac{1}{2}}\geq|\mathcal{D}_k(f,s)|$$

for all $s \geq 1$, then the claimed inequality holds. □

Brocard's problem asks if there are a finite number of solutions to the equation $n!+1=m^2$. Proposition 2.3 provides a helpful inequality to explore. The current trend can be used to investigate a much broader version of the problem. We may improve the outcome by using the **Diagonal** inequality.

Proposition 2.1 (The Diagonal method)**.** *Let*

$$|\mathcal{D}_k(f,s)|-\frac{1}{f(s)}\bigg(\sqrt{\int\limits_{1}^{s}|f'(t)|^2dt}\bigg)\times\bigg(\int\limits_{1}^{s}|\mathcal{D}_k(f,t)|^2dt\bigg)^{\frac{1}{2}}\geq 0$$

for all $s \geq 1$. If

$$\lim_{s\longrightarrow\infty}\bigg(\frac{1}{f(s)}\sum_{n\leq s}f(n)\bigg)\bigg(1-\frac{1}{f(s)}\sqrt{\int\limits_{1}^{s}|f'(t)|^2dt}\bigg)^{-1}<\infty$$

then the equation $f(n)+k=m^2$ has only a finite number of solutions in $\mathbb{N}$ for a fixed $k \in \mathbb{N}$.

Proof. Appealing to Proposition 2.3, it follows under the requirements that $|\mathcal{D}_k(f)|<\infty$, and the claim follows immediately. □

Remark 2.4. The upper bound derived in Proposition 2.3 supplies a somewhat useful tool to study the size of the quantity

$$\#\{n \leq s \mid f(n) + k = m^2\}$$

and in particular Brocard's problem which asks if the set of integers whose factorials are unit left translate of a square is either an infinite set or a finite set. It is worth noting that the upper bounds we have derived do not depend on the size of the shift but on the underlying function. This uniformity does suggest the actual size of the quantity

$$\#\{n \leq s \mid f(n) + k = m^2\}$$

will mostly be influenced by the function under consideration. In various circumstances the ease with which to verify the underlying conditions will inform the category of bounds to exploit. Now we apply the *Diagonal method* to study a slight variant of Brocard's problem.

Lemma 2.5. *The estimate holds*

$$\left(\int\limits_{1}^{s} |\mathcal{D}_k(f,t)|^2 dt\right)^{\frac{1}{2}} \ll |\mathcal{D}_k(f,s)|^{\frac{3}{2}}.$$

Remark 2.6. The upper bound in Lemma 2.5 can easily be obtained by exploiting the methods of integrating a function in elementary calculus.

Definition 2.7 (The r^{th} truncated Gamma function)**.** Let $r \in \mathbb{N}$ be fixed. Then by the r^{th} **truncated** Gamma function Γ_r, we mean the function

$$\Gamma_r(n) := \begin{cases} n(n-1)\cdots(n-r) \ \textbf{if} \ n > r \\ 0 \ \textbf{otherwise}. \end{cases}$$

Lemma 2.8. *For all $s > r$, we have*

$$\frac{1}{\Gamma_r(s)}\sqrt{\int\limits_{1}^{s} |\Gamma_r'(t)|^2 dt} \asymp \frac{1}{s^{\frac{1}{2}}}.$$

Proof. It follows naturally from the definition of the r^{th} **truncated** Gamma function that $\Gamma_r(s) \asymp s^{r+1}$ so that

$$\int\limits_{1}^{s} |\Gamma_r'(t)|^2 dt \asymp s^{2r+1}$$

and the claimed upper bound is an easy consequence. □

Theorem 2.9 (variational Brocard)**.** *The equation $\Gamma_r(s) + k = m^2$ has finitely many solutions $s \in \mathbb{N}$ with $s > r$ for a fixed $k, r \in \mathbb{N}$, where Γ_r is the r^{th} truncated Euler Gamma function.*

Proof. We first apply Lemma 2.8 and notice that

$$\left(\int_1^s |\mathcal{D}_k(\Gamma_r,t)|^2 dt\right)^{\frac{1}{2}} \ll |\mathcal{D}_k(\Gamma_r,s)|^{\frac{3}{2}} \ll |\mathcal{D}_k(\Gamma_r,s)|\sqrt{s}$$

since $|\mathcal{D}_k(\Gamma_r,s)| < s$. It suffices to check that

$$\lim_{s\longrightarrow\infty}\frac{1}{\Gamma_r(s)}\sum_{n\leq s}\Gamma_r(n) < \infty$$

and that

$$\lim_{s\longrightarrow\infty}\frac{1}{\Gamma_r(s)}\sqrt{\int_1^s |\Gamma_r'(t)|^2 dt} < \infty$$

and using the inequality

$$(\int_1^s |\mathcal{D}_k(\Gamma_r,t)|^2 dt)^{\frac{1}{2}} \geq |\mathcal{D}_k(\Gamma_r,s)|.$$

□

3. **Data availability statement**

Manuscript has no associated data.

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