On Primes of the Form 2x-q (where q is a prime less than or equal to x) and the Product of the Distinct Prime Divisors of an Integer (Revised):

A Function Approach to Proving the Goldbach Conjecture by Mathematical Induction

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Abstract. This paper is really an attempt to solve the age-old problem of the Goldbach Conjecture, by restating it in terms of primes of the form 2x-q (where q is a prime less than or equal to x). Restating the problem merely requires us to ask the question: Does a prime of form 2x-q lie in the interval [x, 2x]? We begin by introducing the product, m, of numbers of the form 2x-q. Using the geometric series, an upper bound is estimated for the function m. Next, we prove a theorem that states every even number, 2x, that violates Goldbach's Conjecture must satisfy an inequality involving a simple multiplicative function defined as the product, ρ(m), of the distinct prime divisors of m. A proof of the Goldbach Conjecture is then evident by contradiction as a corollary to the proof of the inequality.

Mathematics Subject Classification: 11N05

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Introduction

This paper is really an attempt to solve the age-old problem of the Goldbach Conjecture, named after the Prussian-born number theorist Christian Goldbach (1690-1764), by restating it in terms of primes of the form 2x-q (where q is a prime less than or equal to x). The problem was represented by Euler as: *every even number greater than 4 is the sum of two primes*. Restating the problem requires us to ask the question: Does a prime of form 2x-q lie in the interval [x, 2x]? This is analogous to the problem of Bertrand's postulate, which asks if a prime number lies between every x and 2x.

To restate the problem of Goldbach's Conjecture we introduce a multiplicative function, which is defined as the product, ρ(y), of the distinct prime divisors of y. The value of the function method as a tool in approaching this sort of problem is highlighted by Rose (see chapter 2 of [Ro]). In the literature there is little evidence of any interest in this function, ρ. An exception is found in the work of Haukkanen and Ruokonen [HR] who employ it in their definition of an exponentially multiplicative function. We may give examples of the values of this function as follows: -

$\rho(1) = 1$
$\rho(2) = 2$
$\rho(3) = 3$
$\rho(4) = 2$
$\rho(5) = 5$
$\rho(6) = 2.3 = 6$
$\rho(7) = 7$
$\rho(8) = 2$
$\rho(9) = 3$
$\rho(10) = 2.5 = 10$
$\rho(11) = 11$
$\rho(12) = 2.3 = 6$
$\rho(13) = 13$
$\rho(14) = 2.7 = 14$
$\rho(15) = 3.5 = 15$

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$\rho(16) = 2$ $\rho(17) = 17$ $\rho(18) = 2.3 = 6$

Of the properties of ρ which are notable, it is brought to the reader's attention that

i. For a prime p, $\rho(p) = p$ and also for a quantity y having only prime factors of power 1, $\rho(y) = y$
ii. Either $\rho(y) = y$ or $\rho(y) < y$
iii. ρ is a multiplicative function

It may be seen readily that ρ is a multiplicative function by considering the function ρ on products m and n, which are co-prime to each other.

Expressing m and n in canonical form we have

$$m = q_1^{a1} .q_2^{a2}. q_3^{a3} \dots .q_s^{as} \quad ,$$

$$ai \geq 0 \text{ , and } q_1<q_2<q_3 <\dots .<q_s$$

$$n = p_1^{b1} .p_2^{b2}. p_3^{b3} \dots .p_r^{br} \quad ,$$

$$bi \geq 0, \ p_1<p_2<p_3 <\dots .<p_r$$

where p_i, q_j are prime. Hence

$$\rho(m) = q_1^{c1} .q_2^{c2}. q_3^{c3} \dots .q_s^{cs} \quad , \qquad ci = 0 \text{ or } 1$$

$$\rho(n) = p_1^{d1} .p_2^{d2}. p_3^{d3} \dots .p_r^{dr} \quad , \qquad di = 0 \text{ or } 1$$

Where ci and di are the exponents of the distinct prime factors of $\rho(m)$ and $\rho(n)$ respectively. Since m and n are co-prime, no p_i occur among the q_j, hence

$$\rho(mn) = (q_1^{c1} .q_2^{c2}. q_3^{c3} \dots .q_s^{cs})(p_1^{d1} .p_2^{d2}. p_3^{d3} \dots .p_r^{dr})$$

$$= \rho(m)\, \rho(n)$$

which meets the condition for the function ρ to be multiplicative.

Having defined ρ we proceed to define an inequality test for even numbers 2x that violate Goldbach's Conjecture. The reader will later discover that the corollary to proving this inequality is the proof of Goldbach's Conjecture. However, before we do this we must estimate the upper bound of a special and novel product function.

Part 1. Estimate of the Upper Bound of a Special Product Function, m

We begin by introducing m, the product of numbers of the form 2x-q (where q is a prime less than or equal to x), which is given as

$$m = \prod_{q \le x} (2x-q)$$

$$= (2x-2)(2x-3)(2x-5)(2x-7). \ldots(2x-q_b) \qquad \textbf{(1)}$$

where the subscript $b = \pi(x)$, the number of primes not exceeding x.

Our approach to the estimation of a valid upper bound for m, first requires us to arrive at an informed guess of the upper bound by representing the situation graphically. We then go on to confirm the upper bound by evaluating logm. This estimate of m is of essential use in completing the proof. We begin by considering m in the form of equation (1): -

$$m = (2x-2)(2x-3)(2x-5)(2x-7). \ldots(2x-q_b),$$

So, divide by $(2x)^{\pi(x)}$: -

$$m/(2x)^{\pi(x)} = (1-(2/2x))(1-(3/2x))(1-(5/2x)) \ldots(1-(q_b/2x))$$

Take logs: -

$$\log(m/(2x)^{\pi(x)}) = \log(1-(2/2x)) + \log(1-(3/2x)) + \log(1-(5/2x)) + \ldots+ \log(1-(q_b/2x)) \quad \textbf{(2)}$$

where log(x) is the natural log of x. Now, if we integrate the geometric series term by term we obtain: -

$$-\log(1-z) = \int(1/(1-z))dz = z + z^2/2 + z^3/3 + z^4/4 +\ldots , \ z < 1 \qquad \textbf{(3)}$$

So for each prime q in (2) we make the term by term substitution of $z = q/2x$ to give (using the expression for log(1-z) in (3)): -

$$-\log(m/(2x)^{\pi(x)}) = [(2/2x) + (1/2)(2/2x)^2 + (1/3)(2/2x)^3 + \ldots] + [(3/2x) + (1/2)(3/2x)^2 + (1/3)(3/2x)^3 + \ldots] + [(5/2x) + (1/2)(5/2x)^2 + (1/3)(5/2x)^3 + \ldots] + \ldots + [[(q_b /2x) + (1/2)(q_b /2x)^2 + (1/3)(q_b /2x)^3 + \ldots]$$

We call this term U. Since $U = -\log(m/(2x)^{\pi(x)})$ it is apparent that the function m can be estimated as

$$m = (2x)^{\pi(x)}e^{-u} \qquad \textbf{(4)}$$

We put U = kx and estimate U and hence kx for values of x up to 400 000.

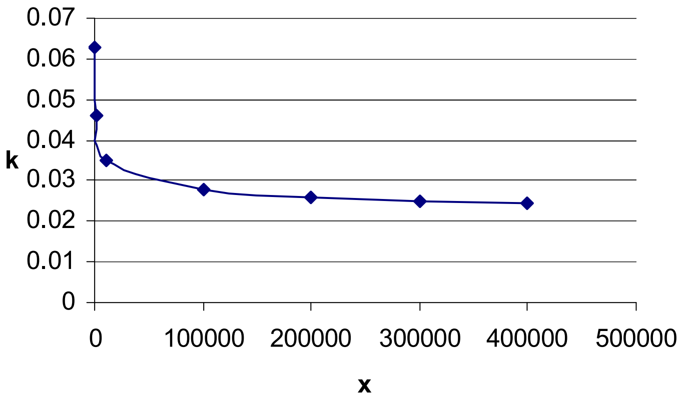
TABLE 1: CONVERGENCE OF k

x	k
100	0.0629
1000	0.0463
10000	0.0350
100000	0.0277
200000	0.0261
300000	0.0252
400000	0.0246

A plot of this table reveals that the value of k converges towards zero. In fact, from the form of the curve, k may be given by

$$k = C/\log x \qquad \textbf{(5)}$$

Fig. 1 Convergence of k



Without, at this point, being concerned whether C is a variable or a constant, the product function m is now given by (4) and (5) as

$$m = (2x)^{\pi(x)} e^{-Cx/\log x} \qquad \textbf{(6)}$$

We may now move on to determine the lower bound of C and this will give us the upper bound of the function m. We do this by first arriving at an informed guess of the lower bound of C by considering a number H such that $x < H < 2x$ and $H^{\pi(x)} = m$ and using this to arrive at a graphical representation of the situation. After this, we confirm the guess at a lower bound, by applying the geometric series once more. So, by (6) we have

$$H^{\pi(x)} = (2x)^{\pi(x)} e^{-Cx/\log x} \qquad \textbf{(7)}$$

But by the Prime number Theorem $\pi(x) \approx x/\log x$ for large x. Using this result (7) becomes

$$H^{x/\log x} \approx (2x)^{x/\log x} e^{-Cx/\log x}$$

$$H \approx (2x)e^{-C}$$

$$C \approx -\log(H/2x) = C^* \qquad \textbf{(8)}$$

In fact, as $x \to \infty$, $(\pi(x)/(x/\log x)) \to 1$ and $C \to -\log(H/2x)$. From (8) we can determine the increasing trend of C* from the calculation of $H = m^{1/\pi(x)}$. It will be shown that for sufficiently large x, that the guess $C > C^* > 0.19$, is correct. Since $C = (\pi(x)/(x/\log x))\, C^*$, we guess that $C > C^* > 0.19$ and this in turn leads us to the following guess for the upper bound of m: -

$$m = (2x)^{\pi(x)} e^{-Cx/\log x} < (2x)^{\pi(x)} e^{-0.19x/\log x} \qquad \textbf{(9)}$$

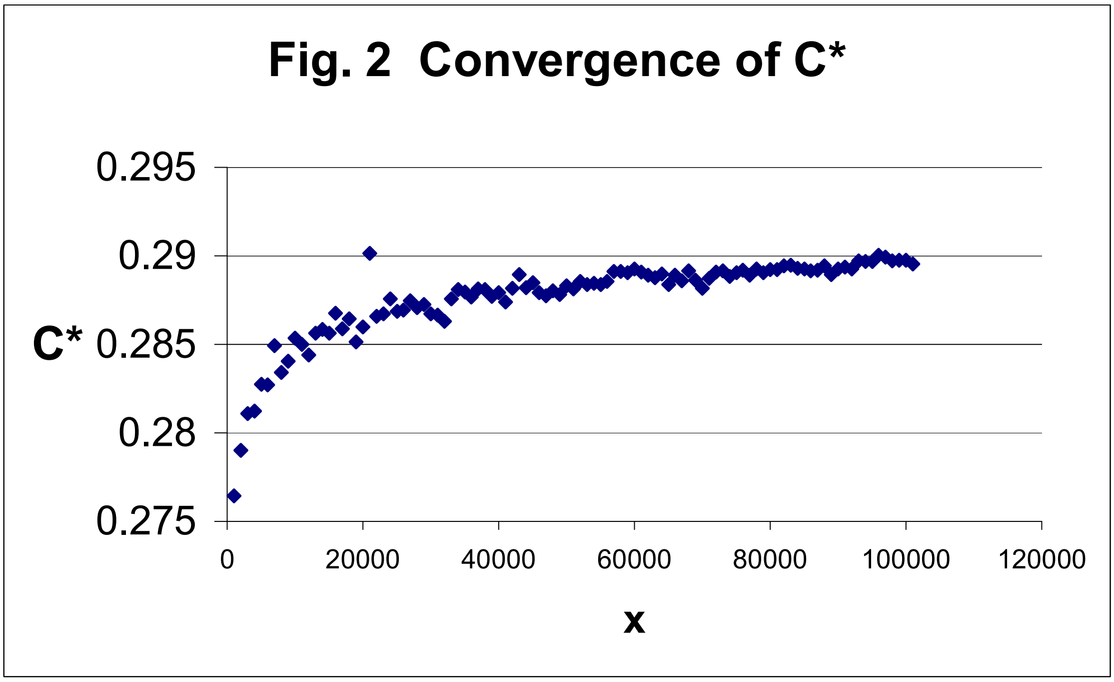
TABLE 2: EXEMPLARY CALCULATIONS OF C*

2x	**log(2x)**	**log(m) = Pi(x)log(H)**	**Pi(x)**	**log(H)**	**H**	**C* = -log(H/2x)**
2000	7.600902	1230.51	168	7.324464	1516.961	0.276438174
4000	8.29405	2428.56	303	8.01505	3026.159	0.279000135
50000	10.81978	29091.87	2762	10.5329	37530.16	0.286878212
100000	11.51293	57615.91	5133	11.22461	74952.32	0.288318023
150000	11.91839	85975.74	7393	11.62934	112346.6	0.289046599
200000	12.20607	114301.24	9592	11.91631	149688.2	0.289763221
250000	12.42922	142441.61	11734	12.13922	187066.6	0.289995982
300000	12.61154	170623.43	13848	12.32116	224394.4	0.290377297
350000	12.76569	198558.61	15916	12.47541	261819.1	0.290279411
400000	12.89922	226750.26	17984	12.60844	299073.4	0.290775653

(In the table π(x) is written as “Pi(x)”)

Let us use a graphical representation to justify this guess, represented by the inequality, (9). A plot of the function C* against x for at least 100 points up to x = 100,000 (fig. 2) appears to show that C* converges, so that the truth of (9) seems feasible, and 0.19 seems quite conservative as a guess of the lower bound for C* and hence C for sufficiently large x, since it would appear that beyond about C* = 0.275 the value of C* continues to increase with x and will never fall as low as this value again. From the curve we notice that the scatter of the points about a curve “trend line” appears to diminish as x increases, so the graphical representation of the problem seems to point to the truth of (9). This can be rigorously confirmed now by a proof that

$$\log m < \pi(x)\log(2x) - 0.19x/\log x \qquad \text{(Taking logs of (9) above)} \qquad \textbf{(10)}$$



To prove the inequality (9), we apply the geometric series to an expression for logm suggested by Richard Hall [Ha]

$$\log m = \sum_{q \le x} \log(2x-q) = \sum_{q \le x} \left\{ \log x + \int_q^x (1/(2x-t))\, dt \right\}$$

$$= \pi(x)\log x + \int_2^x (\pi(t)/(2x-t))\, dt$$

If we divide both the numerator and denominator of the quotient under the integral sign (above) by 2x, we see this is equivalent to

$$\log m = \pi(x)\log x + (1/2x)\int_2^x (\pi(t)/(1-t/2x))\, dt$$

$$= \pi(x)\log x + (1/2x)\int_2^x \pi(t)\{1 + t/2x + (t/2x)^2 + (t/2x)^3 + \ldots\}dt \qquad \textbf{(11)}$$

when we replace the term 1/(1 - t/2x) under the integral sign with the equivalent geometric series. We note that the term $(t/2x) < 1$ since $t \le x$.

To confirm the truth of our guess, represented by the inequality (9), we evaluate logm (as expressed in (11)), by determining the integral and show that the inequality (10) in fact holds true. So we have

x

$$\pi(x)\log x \quad + \quad (1/2x)\int_{2} \pi(t)\{1 + t/2x + (t/2x)^2 + \dots\}dt \quad < \pi(x)\log(2x) - 0.19x/\log x$$

Cancelling the term π(x)logx from both sides it is left to prove that

$$(1/2x)\int_{2}^{x} \pi(t)\{1 + t/2x + (t/2x)^2 + \dots\}dt \quad < \pi(x)\log(2) - 0.19x/\log x \qquad \textbf{(12)}$$

We do this by proving that

$$(1/2x)\int_{2}^{x} \pi(t)\{1 + t/2x + (t/2x)^2 + \dots\}dt \quad < x\log 2/\log x - 0.19x/\log x = \mathbf{0.503\ x/\log x}$$

$$< \pi(x)\log(2) - 0.19x/\log x \qquad \textbf{(13)}$$

If we employ one of Dusart's [Du] upper bounds on π(x), the prime counting function, given by $\pi(x) < (x/\log x)(1 + 1.2762/\log x)$ (for $x > 1$) then it is easy to show that the integral on the far left is less than 0.503 x/logx. The equivalent expression for π(t) is

$$\pi(t) < (t/\log t)(1 + 1.2762/\log t) \quad = (t/\log t)(1 + K/\log t) \qquad \textbf{(14)}$$

For brevity we put K = 1.2762.

So evaluating the resulting upper bound of the above integral using (14), we have

$$(1/2x)\int_{2}^{x} [\, t/\log t + Kt/(\log t)^2\,]\{1 + t/2x + (t/2x)^2 + \dots\}dt$$

$$= (1/2x)\{\ [t^2/2\log t + Kt^2/2(\log t)^2\,]_{2}^{x} + [t^3/3(2x)\log t + Kt^3/3(2x)(\log t)^2\,]_{2}^{x} +$$

$$[t^4/4(2x)^2\log t + Kt^4/4(2x)^2(\log t)^2\,]_{2}^{x} + [t^5/5(2x)^3\log t + Kt^5/5(2x)^3(\log t)^2\,]_{2}^{x} + \dots\ \}$$

$$= (1/2x)\{\ [x^2/2\log x + Kx^2/2(\log x)^2 - 2^2/2\log 2 - K2^2/2(\log 2)^2] +$$

$$[x^3/3(2x)\log x + Kx^3/3(2x)(\log x)^2 - 2^3/3(2x)\log 2 - K2^3/3(2x)(\log 2)^2\ ] +$$

$$[x^4/4(2x)^2\log x + Kx^4/4(2x)^2(\log x)^2 - 2^4/4(2x)^2\log 2 - K2^4/4(2x)^2(\log 2)^2] +$$

$$[x^5/5(2x)^3\log x + Kx^5/5(2x)^3(\log x)^2 - 2^5/5(2x)^3\log 2 - K2^5/5(2x)^3(\log 2)^2] + \dots\ \}$$

$$< (1/2x)\{\ [x^2/2\log x + Kx^2/2(\log x)^2\,] + [x^3/3(2x)\log x + Kx^3/3(2x)(\log x)^2\ ] +$$

$$[x^4/4(2x)^2\log x + Kx^4/4(2x)^2(\log x)^2\,] + [x^5/5(2x)^3\log x + Kx^5/5(2x)^3(\log x)^2\,] + \dots\ \}$$

(if we remove the negative terms on the left hand side of the above inequality and sum only the positive terms)

$= [x/4\log x \ + \ Kx/4(\log x)^2] \ + \ [x/12\log x \ + \ Kx/12(\log x)^2] \ +$

$[x/32\log x \ + \ Kx/32(\log x)^2] \ + [x/80\log x \ + \ Kx/80(\log x)^2] \ + \ldots$

$= (x/\log x)(1 + K/\log x)(1/4 + 1/12 + 1/32 + 1/80 + 1/192 + 1/448 + \ldots) < (½)(x/\log x)$ **(15)**

$= (¼)(x/\log x) + (1/8)(x/\log x) + (1/16)(x/\log x) +$
$(1/32)(x/\log x) + \ldots < 0.503(x/\log x)$ **(16)**

The infinite sequence of positive terms can be seen to be less than $x/2\log x$ in (15) above, for sufficiently large x, as follows: -

Cancelling the factor of $x/\log x$ from both sides of the inequality (15) above, the reader needs to confirm by induction that

$(1 + K/\log x)(1/4 + 1/12 + 1/32 + 1/80 + 1/192 + 1/448 + \ldots) < ½$ **(16a)**

Basis:

The basis is established when x = 120, for instance. We note the geometric pattern of series' above

$(1 + K/\log(120))(1/4 + 1/12 + 1/32 + 1/80 + 1/192 + 1/448 + 1/1024\ldots)$

$= (1 + K/\log(120))(1/(2.2.2^0) + 1/(2.3.2^1) + 1/(2.4.2^2) + 1/(2.5.2^3) + 1/(2.6.2^4) +$

$1/(2.7.2^5) + 1/(2.8.2^6) + 1/(2.9.2^7) + 1/(2.10.2^8) + 1/(2.11.2^9) \ldots)$

$< (1 + K/\log(120))(1/(2.2.2^0) + 1/(2.3.2^1) + 1/(2.4.2^2) + 1/(2.5.2^3) + 1/(2.6.2^4) +$

$1/(2.7.2^5) + 1/(2.8.2^6) + 1/2^8 + 1/2^9 + 1/2^{10} \ldots)$ (comparing term by term)

$= (1 + K/\log(120))(1/(2.2.2^0) + 1/(2.3.2^1) + 1/(2.4.2^2) + 1/(2.5.2^3) + 1/(2.6.2^4) +$

$1/(2.7.2^5) + 1/(2.8.2^6) + 1/2^7) = (1 + 0.2666)(0.3933) = 0.498 < 1/2$

where $1/2^7 = 1/2^8 + 1/2^9 + 1/2^{10} \ldots$ and $K = 1.2762$.

Induction Step

The required result for completion of the inductive proof follows immediately from the hypothesis:

$(1 + K/\log x)(1/4 + 1/12 + 1/32 + \ldots) < 1/2$

So
$(1 + K/\log(x+1))(1/4 + 1/12 + 1/32 + \ldots) < (1 + K/\log x)(1/4 + 1/12 + 1/32 + \ldots) < 1/2$

Hence it is true for all $x \geq 120$ that

$(1 + K/\log x)(1/4 + 1/12 + 1/32 + \ldots) < 1/2$

Returning back to our integral above, which has the left hand side of the inequality in (15) as its upper bound, we complete the proof that

$$(1/2x) \int_2^x [\, t/\log t + Kt/(\log t)^2 \,]\{1 + t/2x + (t/2x)^2 + \ldots\}dt \quad < x/2\log x$$

$$< 0.503x/\log x \qquad \text{(by (16))}$$

so that

$$\log m = \pi(x)\log x + (1/2x) \int_2^x \pi(t)\{1 + t/2x + (t/2x)^2 + \ldots\}dt$$

$$< \pi(x)\log x + (1/2x) \int_2^x [\, t/\log t + Kt/(\log t)^2 \,]\{1 + t/2x + (t/2x)^2 + \ldots\}dt$$

$$< \pi(x)\log x + x/2\log x < \pi(x)\log x + 0.503x/\log x$$

$$= \pi(x)\log x + \log 2(x/\log x) - 0.19x/\log x = \pi(x)\log x + 0.693x/\log x - 0.19x/\log x$$

$$< \pi(x)(\log x + \log 2) - 0.19x/\log x = \pi(x)\log 2x - 0.19x/\log x$$

Hence the upper bound (9) guessed earlier is confirmed to be true

$$m = (2x)^{\pi(x)}e^{-Cx/\log x} < (2x)^{\pi(x)}e^{-0.19x/\log x}$$

Part 2: Goldbach Conjecture Inequality Test

Having determined the upper bound of m we may now proceed by mathematical induction with the proof of an inequality that tests the violation of Goldbach's Conjecture.

THEOREM 2.1

Any even number 2x that violates the Goldbach Conjecture must satisfy the inequality

$$m(x)\,\rho(m(x))_2 \;>\; (2x)^{\pi(x)}e^{-0.19x/\log x}$$

where $\pi(x)$ is the number of primes not exceeding x, and

$$\rho(m) = \rho(m)_1\,\rho(m)_2$$

where $\rho(m)_1$ is the product of the distinct prime divisors of m that are less than x and $\rho(m)_2$ is the product of the distinct prime divisors of m that are greater than or equal to x.

PROOF

We adopt the convention that $\rho(m)_2 = 1$ whenever Goldbach's Conjecture is violated; that is, whenever there are no prime divisors of m greater than or equal to x; such as would occur whenever a prime of form 2x-q (where q is a prime less than or equal to x) does not lie in the interval [x,2x].

In that case, the following relation holds

$$m\,\rho(m)_2 \;=\; m \qquad \textbf{(17)}$$

At this point a small detail on notation is in order. As m is a function of x, we know that $m\,\rho(m)_2$ is also a function of x. We shall recognise this by using the notation $m(x)\rho(m(x))_2$ instead of $m\,\rho(m)_2$ in the induction.

The theorem of mathematical induction states that if P(x) is a statement for each $x \in \mathbf{N}$ (the set of natural numbers) such that

a) $P(x_0)$ is true, and
b) for each $x \in \mathbf{N}$, P(x) is true implies P(x+1) is true,

then P(x) is true for all $x \in \mathbf{N}$

We note that x_0 is the initial value of x and P(x), in our case, is the statement:

$$m(x)\,\rho(m(x))_2 \;>\; (2x)^{\pi(x)}e^{-0.19x/\log x} \qquad \textbf{(18)}$$

This is our *INDUCTION HYPOTHESIS*. Having proved this it would then follow, by (17) above, that $m > (2x)^{\pi(x)}e^{-0.19x/\log x}$, giving the required contradiction with established inequality (9). We proceed with the inductive proof of (18).

BASIS

The basis is easily established for $x \geq 4$

When $x = 4$; $m\rho(m(4))_2 = 150$; $(2x)^{\pi(x)}e^{-0.19x/\log x} = 36.99$
When $x = 5$; $m\rho(m(5))_2 = 9800$; $(2x)^{\pi(x)}e^{-0.19x/\log x} = 554.18$
When $x = 6$; $m\rho(m(6))_2 = 4410$; $(2x)^{\pi(x)}e^{-0.19x/\log x} = 914.59$
When $x = 7$; $m\rho(m(7))_2 = 91476$; $(2x)^{\pi(x)}e^{-0.19x/\log x} = 19394.49$, and so on.

INDUCTION STEP

We have to prove the induction step for the only two possible cases:
case a) $\pi(x+1) = \pi(x)$ and case b) $\pi(x+1) = \pi(x) + 1$.

Case a) $\pi(x+1) = \pi(x)$

We now prove the inductive hypothesis for the first case by first making the assumption that it is true for x.

$$m(x)\rho(m(x))_2 > (2x)^{\pi(x)}e^{-0.19x/\log x}$$

By the main hypothesis, $m\rho(m)_2 = m$. Hence

$$(2x-2)(2x-3)(2x-5)(2x-7).\ \ldots(2x-q_b) > (2x)^{\pi(x)}e^{-0.19x/\log x} \qquad \mathbf{(19)}$$

Now multiply by $Z = ((2(x+1))^{\pi(x+1)}\, e^{-0.19(x+1)/\log(x+1)})/(\,(2x)^{\pi(x)}\, e^{-0.19x/\log x}\,)$ so that

$$Z(2x-2)(2x-3)(2x-5)(2x-7).\ \ldots(2x-q_b) > (2(x+1))^{\pi(x+1)}\, e^{-0.19(x+1)/\log(x+1)}$$

We note that for sufficiently large x

$$(e^{-0.19(x+1)/\log(x+1)}) / (e^{-0.19x/\log x}) = (e^{0.19x/\log x}) / (e^{0.19(x+1)/\log(x+1)}) < 1$$

hence

$$[((2(x+1))^{\pi(x+1)})/(\,(2x)^{\pi(x)})](2x-2)(2x-3)\ldots(2x-q_b) > Z(2x-2)(2x-3)\ldots(2x-q_b) >$$
$$(2(x+1))^{\pi(x+1)}e^{-0.19(x+1)/\log(x+1)}$$

But it may be seen that

$$(2(x+1)-2)(2(x+1)-3)\ldots(2(x+1)-q_b) > [((2(x+1))^{\pi(x+1)})/(\,(2x)^{\pi(x)})](2x-2)(2x-3)\ldots(2x-q_b)$$

To show this we divide by $(2(x+1))^{\pi(x)}$. So we have

$(1-(2/2(x+1)))(1-(3/2(x+1)))\ldots(1-(q_b/2(x+1))) > (1-(2/2x))(1-(3/2x))\ldots(1-(q_b/2x))$

This is readily seen to be true hence

$(2(x+1)-2)(2(x+1)-3)\ldots(2(x+1)-q_b) > (2(x+1))^{\pi(x+1)}e^{-0.19(x+1)/\log(x+1)}$

Now, $(2(x+1)-2)(2(x+1)-3)\ldots(2(x+1)-q_b) = m(x+1)\rho(m(x+1))_2$, because the primes involved in the expression $(2(x+1)-2)(2(x+1)-3)\ldots(2(x+1)-q_b)$ are precisely those in the expression $(2x-2)(2x-3)\ldots(2x-q_b)$. This is due to the fact that $\pi(x+1) = \pi(x)$. Thus

$m(x+1)\,\rho(m(x+1))_2 > (2(x+1))^{\pi(x+1)}e^{-0.19(x+1)/\log(x+1)}$

where the subscript (x+1) denotes a function with respect to (x+1) that is precisely $m\,\rho(m)_2$, when (x+1) is replaced by x. The example of the proof by induction of $1/(1 - 1/x) < x$, for instance, supports the reasoning behind this proof. The final steps of the two proofs are analogous. In the above proof, we arrive at

$m(x+1)\rho(m(x+1))_2 > (2(x+1))^{\pi(x+1)}e^{-0.19(x+1)/\log(x+1)}$

from the penultimate step

$(2(x+1)-2)(2(x+1)-3)\ldots(2(x+1)-q_b) > (2(x+1))^{\pi(x+1)}e^{-0.19(x+1)/\log(x+1)}$

Case b) π(x+1) = π(x) + 1

Again we begin with the induction hypothesis

$$m(x)\,\rho(m(x))_2 > (2x)^{\pi(x)}e^{-0.19x/\log x}$$

Then by the main hypothesis and induction hypothesis

$(2x-2)(2x-3)\ldots(2x-q_b) > (2x)^{\pi(x)}\,e^{-0.19x/\log x}$

Multiply by $W = ((2(x+1))^{(\pi(x)+1)}\,e^{-0.19(x+1)/\log(x+1)})/(\,(2x)^{\pi(x)}\,e^{-0.19x/\log x}\,)$. So

$W(2x-2)(2x-3)(2x-5)(2x-7)\ldots(2x-q_b) > (2(x+1))^{(\pi(x)+1)}\,e^{-0.19(x+1)/\log(x+1)}$

We note that for sufficiently large x

$$(e^{-0.19(x+1)/\log(x+1)}) / (e^{-0.19x/\log x}) = (e^{0.19x/\log x}) / (e^{0.19(x+1)/\log(x+1)}) < 1$$

hence

$[((2(x+1))^{(\pi(x)+1)})/(\,(2x)^{\pi(x)})](2x-2)(2x-3)\ldots(2x-q_b) > W(2x-2)(2x-3)\ldots(2x-q_b) >$

$(2(x+1))^{(\pi(x)+1)}\, e^{-0.19(x+1)/\log(x+1)}$

But it may be shown that

$$m(x+1)\rho(m(x+1))_2 \;>\; [((2(x+1))^{(\pi(x)+1)})/(\;(2x)^{\pi(x)})](2x-2)(2x-3)\ldots(2x-q_b)$$

Because x+1 is prime, then $\rho(m(x+1))_2 \geq x+1$, by definition. Hence,

$m(x+1)\rho(m(x+1))_2 \;\geq (x+1)(2(x+1)-2)(2(x+1)-3)\ldots(2(x+1)-q_b)(2(x+1)-q_{b+1})$

Now

$(x+1)(2(x+1)-2)(2(x+1)-3)\ldots(2(x+1)-q_b)(2(x+1)-q_{b+1}) >$

$[((2(x+1))^{(\pi(x)+1)})/(\;(2x)^{\pi(x)})](2x-2)(2x-3)\ldots(2x-q_b)$

To see this we divide by $(2(x+1))^{(\pi(x)+1)}$ so that

$(x+1)(1-(2/2(x+1)))(1-(3/2(x+1)))\ldots(1-(\,q_b/2(x+1)))\,(1-(\,q_{b+1}/2(x+1)))\;) >$

$(1-(2/2x))(1-(3/2x))\ldots(1-(\,q_b/2x))$

But note that q_{b+1} is the prime, x+1, so that $(1-(\,q_{b+1}/2(x+1)))\;) = 1/2$, so

$(1/2)(x+1)(1-(2/2(x+1)))(1-(3/2(x+1)))\ldots(1-(\,q_b/2(x+1)))\;) >$

$(1-(2/2x))(1-(3/2x))\ldots(1-(\,q_b/2x))$

which is readily seen to be true since $(1-(2/2(x+1)))(1-(3/2(x+1)))\ldots(1-(\,q_b/2(x+1)))\;) >$ $(1-(2/2x))(1-(3/2x))\ldots(1-(\,q_b/2x))$.

So it follows that

$m(x+1)\rho(m(x+1))_2 \geq (x+1)(2(x+1)-2)(2(x+1)-3)\ldots(2(x+1)-q_b)(2(x+1)-q_{b+1})$

$> \;\; [((2(x+1))^{(\pi(x)+1)})/(\;(2x)^{\pi(x)})](2x-2)(2x-3)\ldots(2x-q_b)$

$> \; W(2x-2)(2x-3)\ldots(2x-q_b)$

$> \; (2(x+1))^{(\pi(x)+1)}\, e^{-0.19(x+1)/\log(x+1)}$

as required.

COROLLARY –
Proof of Theorem 2.1 Supports the Truth of the Goldbach's Conjecture

Any even number 2x that violates Goldbach's Conjecture will also satisfy the condition that it's closed interval [x, 2x] contains no primes of the form 2x-q where q is a prime less than or equal to x; since satisfying this condition ensures that 2x cannot be expressed as a sum of two primes. The fact is established by analysis in Part 1 that m(x) is bounded above by the same value as its lower bound in the inequality (18). We recall that

$$m(x) < (2x)^{\pi(x)} e^{-0.19x/\log x} \qquad \ldots\ldots\ldots\ldots(9)$$

The contradiction then follows on from the fact that any interval [x, 2x] which contains no primes of form 2x-q where q is a prime less than or equal to x, will satisfy the relation $m(x)\,\rho(m(x))_2 = m(x)$. Since then $\rho(m(x))_2 = 1$. So by inequality (18)

$$m(x)\,\rho(m(x))_2 = m(x) > (2x)^{\pi(x)} e^{-0.19x/\log x}$$

Thus m(x) is bounded above by the same value in (9) as it's lower bound established by the above inequality (18). This contradiction supports the truth of Goldbach's Conjecture. This is true, even though the basis for the induction hypothesis (16a): $(1 + K/\log x)(1/4 + 1/12 + 1/32 + \ldots) < ½$, is not established until $x \geq 120$; since Goldbach's Conjecture is true for $4 \leq x < 120$.

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