Robust Extreme Quantile Estimation for Pareto-Type tails through an Exponential Regression Model

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ABSTRACT

The estimation of extreme quantiles is one of the main objectives of statistics of extremes (which deals with the estimation of rare events). In this paper, a robust estimator of extreme quantile of a heavy-tailed distribution is considered. The estimator is obtained through the minimum density power divergence criterion on an exponential regression model. The proposed estimator was compared with two estimators of extreme quantiles in the literature in a simulation study. The results show that the proposed estimator is stable to the choice of the number of top order statistics and show lesser bias and mean square error compared to the existing extreme quantile estimators. Practical application of the proposed estimator is illustrated with data from the pedochemical and insurance industries.

Key Words: Extreme quantile; robust estimation; exponential regression model; minimum density power divergence.

Mathematical Subject Classification: 62G05, 62G32, 62G35.

1 Introduction

In statistics of extremes, one of the main objectives is the estimation of extreme quantiles, usually, beyond the range of available data. Important practical applications include the estimations of Value-at-Risk (Gilli and Këllezi 2006), extreme earthquake (Pisarenko and Sornette 2003), height of hydroelectric dam (Minkah 2016), height of sea dikes (de Haan 1990) and insurance losses (Peng and Qi 2006), among many others.

When outliers and deviations from assumed parametric models are present, it has been documented in Dell'Aquila and Embrechts (2006) and Vandewalle et al. (2007) that robust statistics provides a better method of reducing mean square errors and biases although the concepts seems to be contradictory at first sight. In addition, the concept of the minimum density power divergence (MDPD) of Basu et al. (1998) has been shown to provide robust estimators of the tail index when contaminations are present within heavy-tailed sample observations (see e.g. Juarez and Schucany 2004; Kim and Lee 2008; Dierckx et al. 2013; Ghosh 2017; Minkah et al. 2021).

However, not much attention has been given to the estimation of high quantiles, small exceedance probabilities and return periods which are the focal points of statistics of extremes. An exception to this is the work of Goegebeur et al. (2014) where the authors introduced a robust and asymptotically unbiased extreme quantile estimator using the minimum density power divergence estimator (MDPDE) of the tail index in Dierckx et al. (2013). The estimation of the tail index and subsequent extreme quantile estimator were based on relative excesses. The resulting quantile estimator was shown to be better in terms of MSE than the extreme quantile estimator based on the maximum likelihood estimator (MLE) of the tail index and the Weismann-type estimator obtained from the Hill estimator of the tail index, particularly in the presence of contamination in the sample.

In the present paper, we propose an alternative robust extreme quantile estimator based on the MDPDE of the tail index from Minkah et al. (2021). Our approach is different from that of Goegebeur et al. (2014) in that our estimator is based on log-spacings of order statistics instead of relative excesses. The robust estimation of the tail index, as developed in Minkah et al. (2021), simultaneously yields estimators of second-order parameters to generate new reduced-bias estimators of high quantilesThe motivation for the usage of this estimator arises from the good performance of the MDPDE of the tail index, as developed in Minkah et al. (2021), compared with the estimators from Dierckx et al. (2013) and others.

The rest of the paper is organised into four sections. In Section 2, we present a review of robust estimation of the tail index and the proposed extreme quantile estimators. In Section 3, a simulation study that compares the proposed estimator to that of Goegebeur et al. (2014) and the Weissman-type using the Hill estimator are presented and discussed. Section 4 presents an application of the proposed estimator to the estimation of extreme quantiles of data sets from insurance and pedochemical studies. Lastly, Section 5 provides concluding remarks.

2 Robust Estimation of the Tail Index and Extreme Quantiles

Let $X_1, X_2, \ldots, X_n$ be a sample of independent and identically distributed (i.i.d) observations with underlying distribution F and let $X_{1,n} \leq X_{2,n} \leq \ldots \leq X_{n,n}$ denote the associated order statistics. In the extreme value framework, F belongs to one of the three domains of attraction viz. Gumbel (light-tail), Weibull (short tail with possible finite right end-point) and Fréchet (heavy-tail).

In this paper, we consider F as a distribution in the Fréchet domain of attraction with survival function of the form,

$$1 - F(x) = x^{-1/\gamma} \ell_F(x), \tag{1}$$

and the tail quantile function given by,

$$U(x) = Q(1 - 1/x) = x^{\gamma} \ell_U(x), \tag{2}$$

where $\gamma > 0$ is the tail index, Q is the quantile function of F, and ℓ_F and ℓ_U are slowly varying functions satisfying respectively,

$$\lim_{x \to \infty} \frac{\ell_F(xt)}{\ell_F(x)} = 1, \ \forall t > 0, \tag{3}$$

and

$$\lim_{x \to \infty} \frac{\ell_U(xt)}{\ell_U(x)} = 1, \ \forall t > 0. \tag{4}$$

The aim of this paper is to provide a robust estimator of extreme quantiles, $x_{p_n} := F^{-1}(1-p_n)$, where $p_n \to 0$. We are therefore interested in the estimation of

$$x_{p_n} = F^{-1}(1-p_n) = Q(1-p_n) = U(1/p_n). \tag{5}$$

For this, a brief description of the procedure for robust estimation of the required parameters is described in Section 2.1 and the proposed extreme quantile estimation method is then developed in Section 2.2.

2.1 Estimation of Tail Index

Minkah et al. (2021) used the idea from Beirlant et al. (1999, pg. 183) with an appropriate second-order assumption on the slowly varying function ℓ under which the weighted log-spacings of the order statistics, namely,

$$Z_i = i.\left(\log X_{n-i+1,n} - \log X_{n-i,n}\right), \quad 1 \le i \le k \le n, \tag{6}$$

are shown to be approximately exponentially distributed i.e.

$$Z_i \sim \left(\gamma + b_{n,k}\left(\frac{i}{k+1}\right)^{-\rho}\right) E_i, \tag{7}$$

where $E_1, \ldots, E_n$ are independent and identically distributed random variables each having a standard exponential distribution, $b_{n,k}$ $(b_{n,k} = b\left((n+1)/(k+1)\right) \to 0$ as $k,\ n \to \infty)$ and $\rho < 0$, are the second-order parameters. It should be noted that Z_i's are independent having approximate density f_{θ_i}, an exponential density with mean $\theta_i = \gamma + b_{n,k}\left(\frac{i}{k+1}\right)^{-\rho}$. However, the identically distributed assumption does not hold.

Minkah et al. (2021) proposed to estimate the parameters in (7) by using the minimum density power divergence method of Basu et al. (1998) and Ghosh and Basu (2013). Thus, the estimators of the parameters, γ and $b_{n,k}$, are obtained by minimising the divergence between the data and the model density, or equivalently the objective function (Minkah et al. 2021),

$$H_k(\gamma, b, \rho) = \frac{1}{k-1}\sum_{i=1}^{k-1}\left[\frac{1}{(1+\alpha)\theta_i^{\alpha}} - \frac{1+\alpha}{\alpha\theta_i^{\alpha}}\exp\left(-\frac{\alpha z_i}{\theta_i}\right)\right] \tag{8}$$

where $\theta_i = \gamma + b\left(\frac{i}{k+1}\right)^{-\rho}$ with $b = b_{n,k}$. Note that in their proposal the parameter ρ is estimated externally (See e.g. Fraga Alves and de Haan (2003); Gomes and Rodrigues (2008) for some estimators of ρ.).

In similar work preceding Minkah et al. (2021), Dierckx et al. (2013) also proposed robust estimators of the parameters of the extended Pareto distribution with survival function

$$1 - G(y) = y^{-1/\gamma}(1 + \delta - \delta y^{\tau})^{-1/\gamma}\mathbb{I}\{y > 1\} \tag{9}$$

where, $\gamma > 0,\ \tau < 0$ and $\delta \in \max\{-1,\ 1/\tau\}$. The distribution (9) is fitted to relative excesses over a threshold, $X_{n-k,n}$, denoted $Y_j := X_{n-j+1,n}/X_{n-k,n},\ j = 1, 2, \ldots, k$. The estimation of these parameters was then obtained through the minimum density power divergence criterion of Basu et al. (1998). However, it has been shown that the estimator of the tail index γ has significantly improved the finite-sample properties obtained under the Minkah et al. (2021) approach compared to those obtained under the Dierckx et al. (2013) approach.

2.2 Estimation of Extreme Quantiles

Due to the stable performance of the MDPDE of the tail index in Dierckx et al. (2013) under contaminated samples, Goegebeur et al. (2014) used the estimators of the tail index and the other second-order parameters of the extended Pareto distribution, (9), to introduce a corresponding extreme quantile estimator. This extreme quantile estimator was shown to be robust and asymptotically unbiased. In addition, its finite sample properties show stable results compared to other competing estimators when samples are contaminated with very large observations.

Following the preceding idea, we propose a new quantile estimator based on the exponential regression model and minimum density power divergence criterion of Basu et al. (1998) and Ghosh and Basu (2013) that was used in Minkah et al. (2021). As described in the preceding subsection, Minkah et al. (2021) obtained a robust estimators of the tail index and other second-order parameters of the exponential regression model. These estimators are incorporated into the non-robust extreme quantile estimator of Beirlant and Matthys (2001) for the case, $\gamma > 0$. Thus, the proposed extreme quantile estimator incorporates the MDPDEs of the tail index and the second-order parameters to obtain a reduced-bias estimator that is robust for quantile estimation based on the distribution of the log-spacings of order statistics in (7). The proposed estimator of the extreme quantile $Q(1-p_n)$ is given by,

$$\hat{Q}_{ERM\_Qp}(1-p_n) = X_{n-k,n}\left(\frac{(n+1)p_n}{k+1}\right)^{-\hat{\gamma}} \exp\left(\hat{b}_{n,k}\frac{1-\left(\frac{(n+1)(1-p_n)}{k+1}\right)^{-\hat{\rho}_k}}{-\hat{\rho}_k}\right), \tag{10}$$

where $\hat{\gamma}$ and $\hat{b}_{n,k}$ are the MDPDE of γ and $b_{n,k}$ respectively from (7). In this case, the estimation is done using the procedure in Minkah et al. (2021) for γ, and $b_{n,k}$. $\hat{\rho}_k$ ithe external estimated value of rho-sub-k obtained from using the procedure in Fraga Alves and de Haan (2003) using $k \in \{5,\ldots,n-1\}$.

3 Simulation Study

In this section, we compare the performance of the proposed quantile estimator with the equivalent minimum density power divergence quantile estimators of the Pareto-type tail index in the literature. Specifically, the proposed exponential regression model estimator based on log-spacings of order statistics, ERM_Qp, the Dierckx et al. (2013) estimator obtained from fitting an extended Pareto distribution to relative excesses, EPD_Qp, and the Weissman-type estimator using the Hill estimator, Hill_Qp which is given by,

$$\hat{Q}_{Hill\_Qp}(1-p_n) = X_{n-k,n}\left(\frac{(n+1)p_n}{k+1}\right)^{-\hat{\gamma}}, \tag{11}$$

are compared.

3.1 Simulation Design

We consider three distributions namely, Burr, Pareto and Fréchet, which are in the Fréchet domain of attraction. Each sample generated from a distribution is contaminated with some large observations. To do this, the contamination observations are generated from an additional distribution G, to obtain a mixture distribution with F, such that the mixture contaminated model is: $(1-\varepsilon)F+\varepsilon G$

where ε is a small percentage of the sample size n of observations from F. The distribution of G is chosen such that its parameters produce sample observations that are extremely large compared to those from F. Specifically, G is chosen to have larger scale and tail index than that of F. Table 1 shows the survival function, tail index and the parameter values for each distribution used in the simulation study.

Table 1: Types of Distributions

Distribution	$1-F(x)$	γ	Parameter Values
Burr	$(\eta/(\eta+x^{\tau}))^{\lambda}$, $x>0, \eta, \tau, \lambda>0$	$1/(\tau\lambda)$	$\eta=1, \tau=1/2,\ \lambda=4$; $\eta=1, \tau=1/2,\ \lambda=2$; $\eta=1, \tau=2,\ \lambda=5$.
Fréchet	$1-\exp\left(-[(x-a)/b]^{-\omega}\right)$, $x>0,\ \beta>0,\ x>a,\ b>0$	$1/\omega$	$\omega=1,\ 2,\ 10$.
Pareto	$(x/s)^{-\omega}$, $x>s,\ \omega>0$	$1/\omega$	$\omega=1,\ 2,\ 10$.

Different sizes of the tail index are considered under each distribution: the parameter values are chosen to obtain tail indexes: 0.1 (small), 0.5 (medium) and 1.0 (large). The robustness of the extreme quantile estimators under different contamination scenarios are assessed for $\varepsilon=0.02$ and $\varepsilon=0.05$. Furthermore, the robustness parameter, α, is taken at $0.1,\ 0.5$ and 1, to assess the effect of it on the extreme quantile estimators. The order of these values represent increasing levels of robustness.

In the next sub-section, we present the results for the estimation of extreme quantiles of samples generated from the three distributions.

3.2 Discussion of Simulation Results

The simulation results for the estimation of extreme quantiles are discussed in this section. For ease of presentation, we present the results of the Burr distribution, giving those for the Fréchet and Pareto distributions in the Appendices.

Figures 1-2 show the results of the extreme quantile estimation for the Burr distribution for the tail index values 0.1, 0.5 and 1 respectively. For each plot, it can be seen that the average values of the extreme quantile estimates of the proposed estimator, ERM_Qp and the counterpart EPD_Qp are quite closer to the parameter value of the quantile (represented by the horizontal line in each graph). The middle column in each panel also shows the absolute bias of the estimators. The results show that the proposed estimator has better bias than the competing estimators in most of its trajectory along the number of upper order statistics, k. Similar performance can be seen in relation to the MSEs which are shown in the rightmost column. Furthermore, in the case of the robustness parameter, α, the sample paths become smoother for low values and exhibit more variation for larger values.

With regards to the Pareto distribution, the results are shown in Figures 5-6 in the appendix. The Hill estimator performs poorly and compare to the two MDPDEs, ERM_Qp and EPD_Qp, of extreme quantiles. However, whereas the EPD_Qp performs better under small contamination in Figure 5, the proposed estimator ERM_Qp usually has the least bias and MSE for the larger contamination percentage in Figure 6 especially as k increases, i.e. where bias is prevalent.

In the case of the Fréchet distribution, as shown in Figures 7-8 the EPD_Qp estimator mostly shows less bias and MSE for smaller values of k whereas the proposed ERM_Qp performs better for larger values of k. Again, the Weismann-type estimator performs abysmally and is not shown within most regions of the graphs of bias and MSE.

Therefore, it can be concluded that the proposed estimator, ERM_Qp, is less sensitive to the choice of k and exhibits less bias and lower MSE. Moreover, it is robust to contamination of underlying distribution samples.

4 Application

In this section, we illustrate the application of the proposed extreme quantile estimator on two practical datasets that were studied in Minkah et al. (2021). In that paper, the authors illustrated

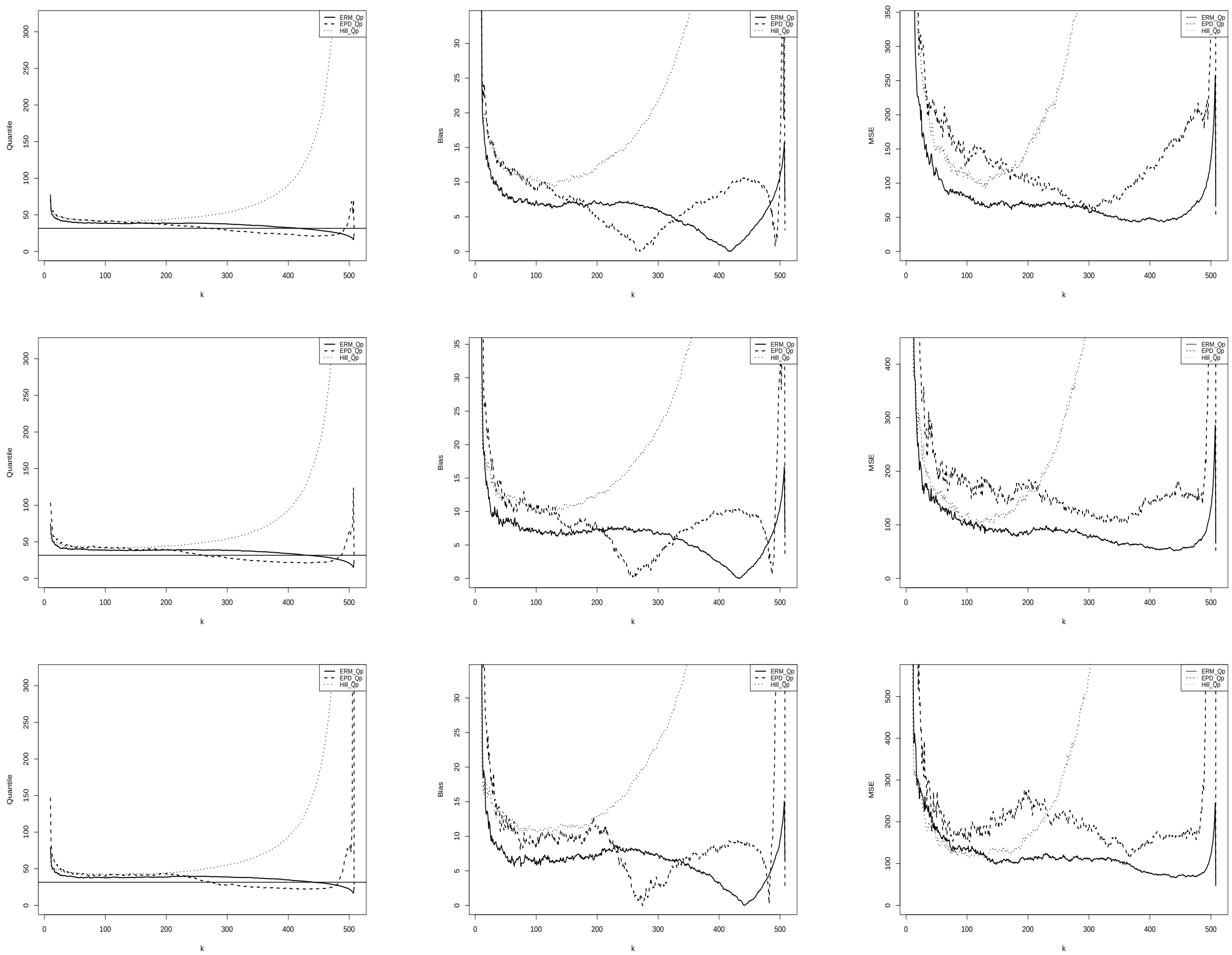

Figure 1: Burr distribution with $n = 500$, $p_n = 0.001$, $\varepsilon = 2\%$ and $\gamma = 0.5$. Topmost row: $\alpha = 0.1$; Middlemost row: $\alpha = 0.5$; and bottommost row: $\alpha = 1.0$. Leftmost column: average of $\hat{Q}(1-p_n)$; middlemost column: Bias; and rightmost column: MSE.

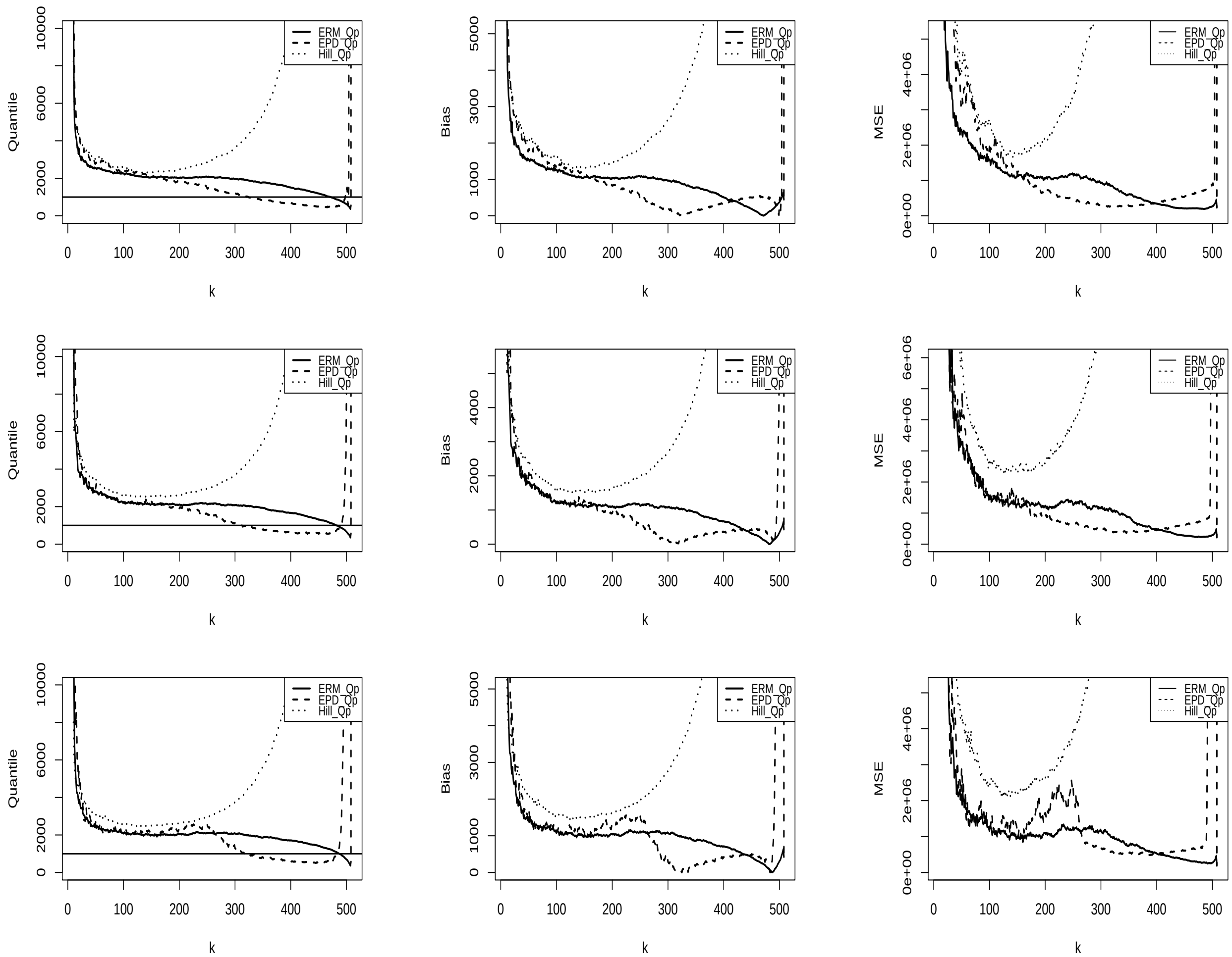

Figure 2: Burr distribution with $n = 500$, $p_n = 0.001$, $\varepsilon = 10\%$ and $\gamma = 1$. Topmost row: $\alpha = 0.1$; Middlemost row: $\alpha = 0.5$; and bottommost row: $\alpha = 1.0$. Leftmost column: average of $\hat{Q}(1 - p_n)$; middlemost column: Bias; and rightmost column: MSE.

the estimation of the tail indexes and in the present paper, we proceed to estimate extreme quantiles.

Figure 3 shows the scatter, exponential and Pareto Quantile-Quantile (QQ) plots in the topmost, middlemost and the rightmost panels respectively. Clearly, some few observations are detached from the bulk of the datasets in both cases of the scatter plots. In addition, these observations deviate from the linearity of the exponential QQ plots and detached from the Pareto QQ plots. These give an indication that these observations are outliers. Moreover, the concave nature of the exponential QQ plots and the near linearity of the Pareto QQ plots strongly suggest that the underlying distributions are in the Fréchet domain of attraction. Therefore, these datasets fit appropriately for the illustration of the proposed estimator of extreme quantiles.

Figure 4 shows the tail index and the corresponding extreme quantile estimators of the underlying distributions of the two datasets. The tail indexes are ERM_M (Minkah et al. 2021), EPD_D (Dierckx et al. 2013) and Hill (Hill 1975) with their respective extreme quantile estimators ERM_Qp, EPD_Qp and Hill_Qp. The ERM_M estimator shows stable estimates of γ and it is also less sensitive to the choice of k (see also Minkah et al. 2021). In addition, the proposed ERM_Qp inherits these observations as k increases. Thus, it exhibits a desirable property of extreme quantile estimator, i.e. less bias, compared with the Weissman-type estimator (based on the Hill estimator of γ) and the MDPD estimator of Goegebeur et al. (2014). Therefore, the proposed estimator ERM_Qp provides a desirable reduced-bias estimator for extreme quantiles which is one of the main objectives in extreme value analysis.

5 Conclusion

In this paper, a reduced-bias estimator for the estimation of extreme quantiles is proposed. The estimator is based on the exponential regression model fitted to log-spacings of order statistics under the minimum density power divergence approach in the estimation of the tail index. The proposed estimator was compared with two existing estimators of extreme quantiles in the literature. First, the Weissman-type estimator based on Hill estimator of the tail index. Second, a quantile estimator using the minimum density power divergence but with an extended Pareto fitted to the relative

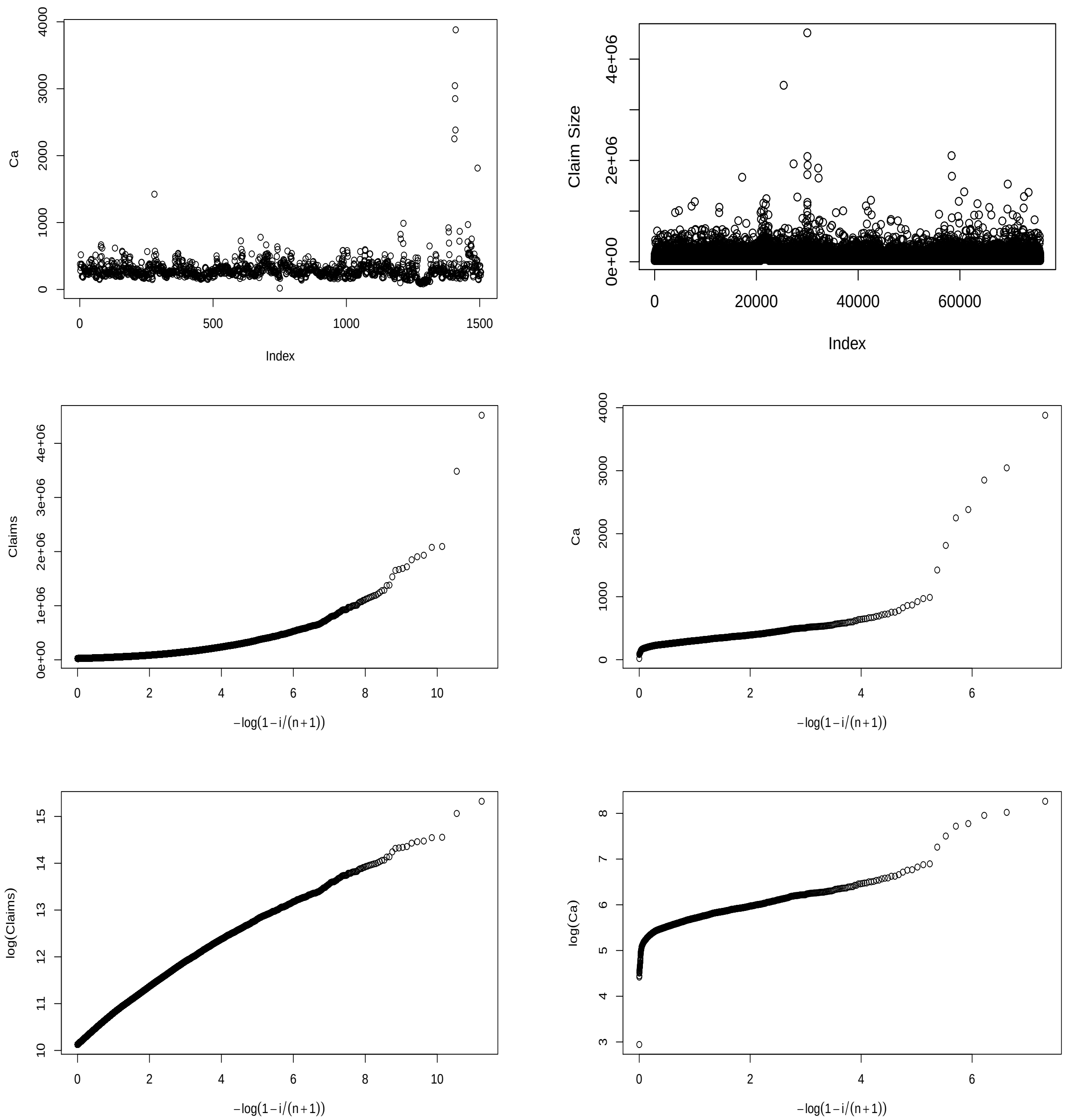

Figure 3: Left panel: SOA data; Right panel: Condroz data

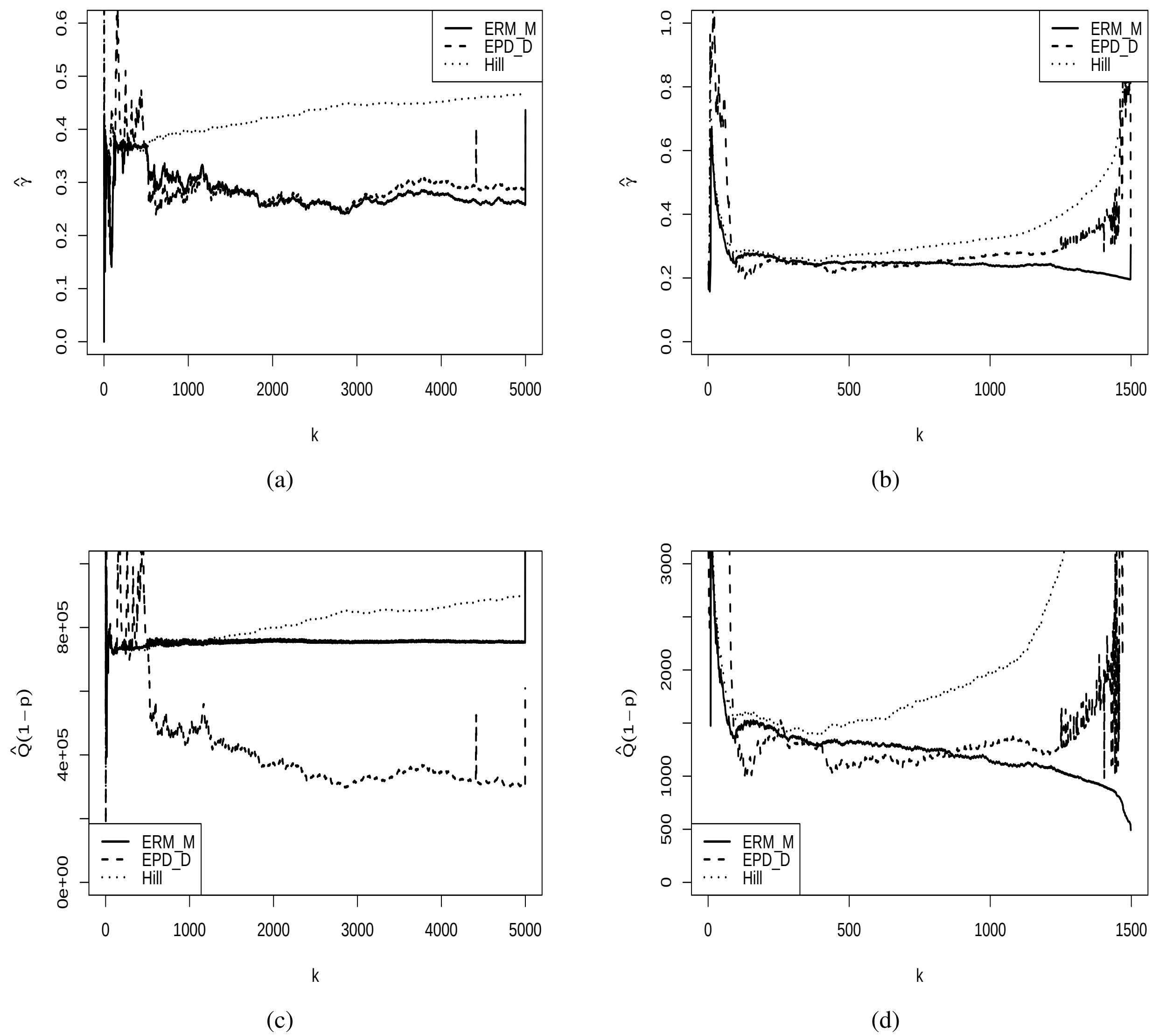

Figure 4: Tail index and Quantile estimators: Condroz data, left panel. SOA data, right panel; γ estimates with $\alpha = 0.2$, top row; $\hat{Q}(0.999)$, bottom row.

exceedances. The results show that the proposed estimator is stable to the choice of the top order statistics and very competitive to the existing robust estimator of extreme quantiles in Dierckx et al. (2013). In addition, it has less bias and lower mean square error in a number of important cases such as the inclusion of intermediate observations. Furthermore, a practical application is demonstrated on the estimation of extreme quantiles in the pedochemical and insurance settings. Similar to quantile estimation, other primary objectives in statistics of extremes such as small exceedance probabilities and return periods can be obtained using this approach. These two objectives and the asymptotic properties of the proposed proposed robust extreme quantile estimator are the subject of future research.

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Disclosure statement

The authors report there are no competing interests to declare

Pareto Distribution

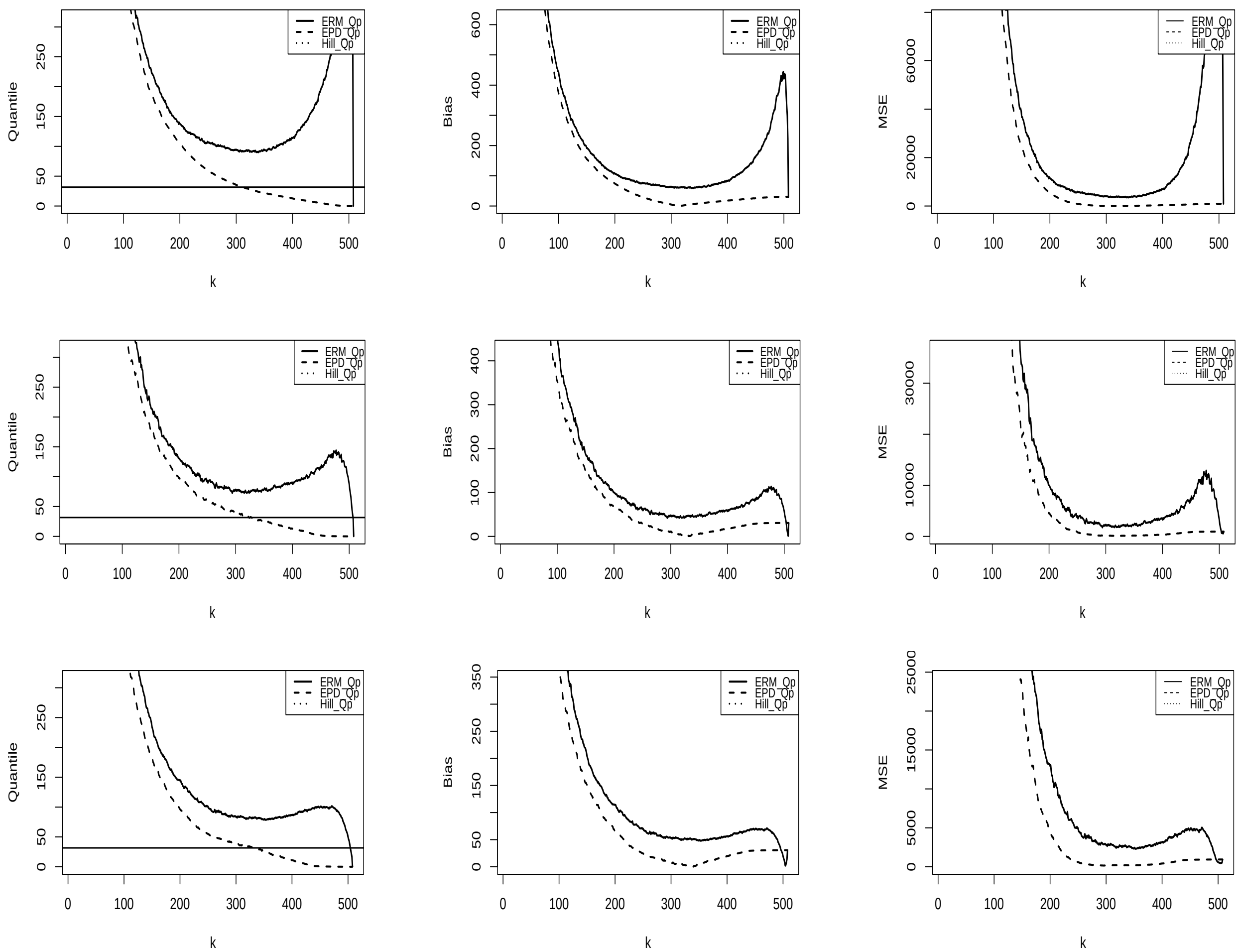

Figure 5: Pareto distribution with $n = 500$, $p_n = 0.001$, $\varepsilon = 2\%$ and $\gamma = 0.5$. Topmost row: $\alpha = 0.1$; Middlemost row: $\alpha = 0.5$; and bottommost row: $\alpha = 1.0$. Leftmost column: average of $\hat{Q}(1 - p_n)$; middlemost column: Bias; and rightmost column: MSE.

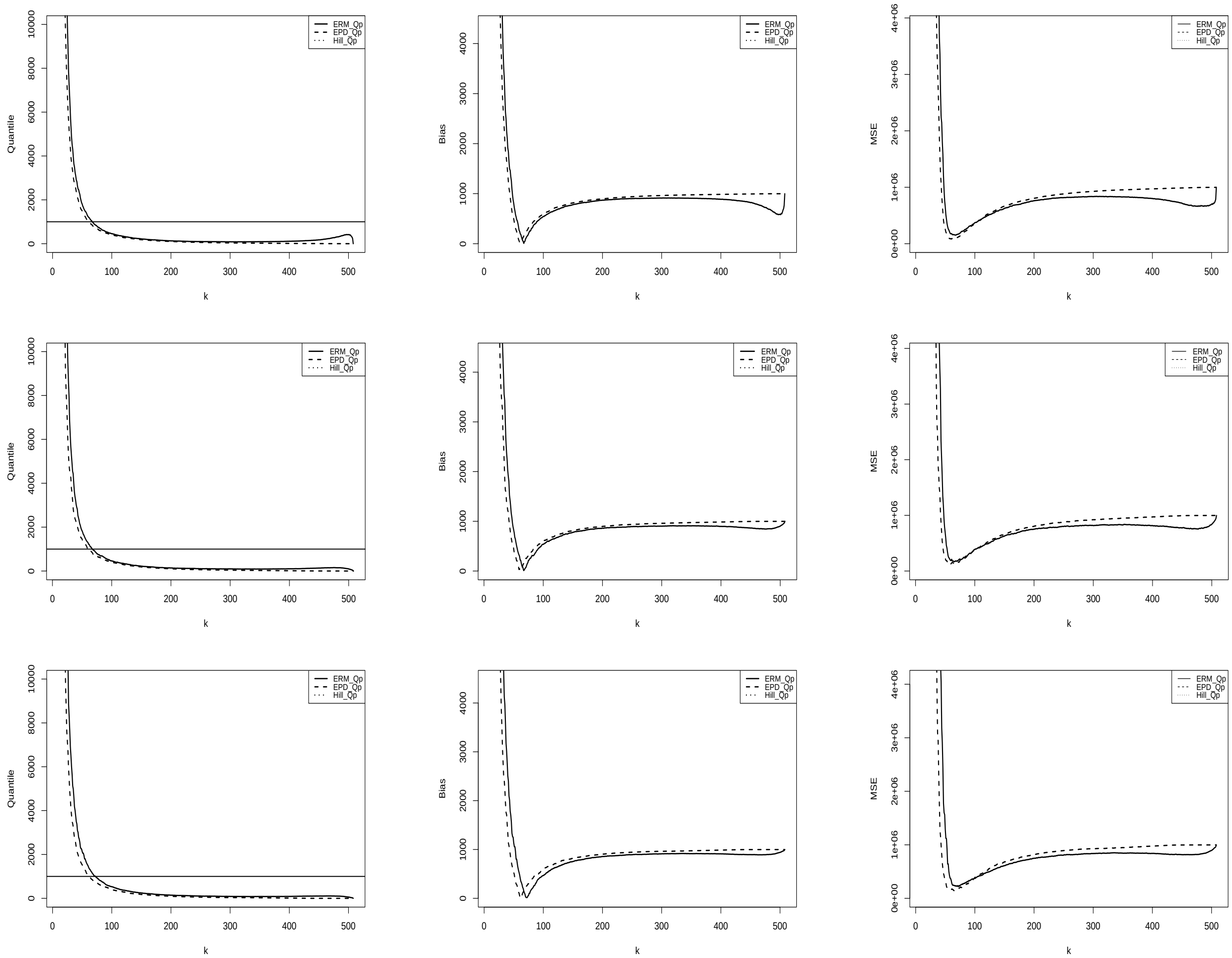

Figure 6: Pareto distribution with $n = 500$, $p_n = 0.001$, $\varepsilon = 10\%$ and $\gamma = 1$. Topmost row: $\alpha = 0.1$; Middlemost row: $\alpha = 0.5$; and bottommost row: $\alpha = 1.0$. Leftmost column: average of $\hat{Q}(1 - p_n)$; middlemost column: Bias; and rightmost column: MSE.

Fréchet Distribution

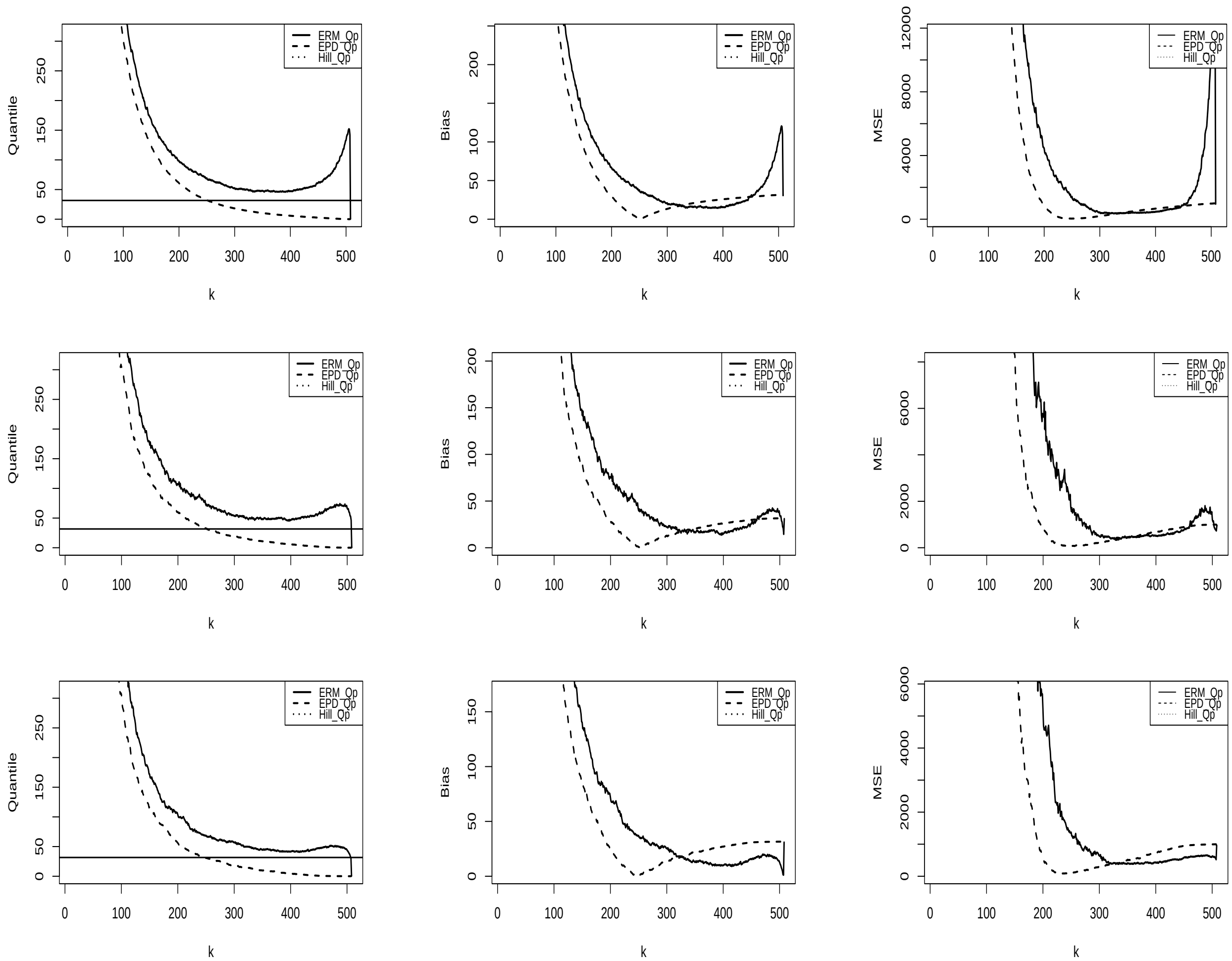

Figure 7: Fréchet distribution with $n = 500,\ p_n = 0.001,\ \varepsilon = 2\%$ and $\gamma = 0.5$. Topmost row: $\alpha = 0.1$; Middlemost row: $\alpha = 0.5$; and bottommost row: $\alpha = 1.0$. Leftmost column: average of $\hat{Q}(1-p_n)$; middlemost column: Bias; and rightmost column: MSE.

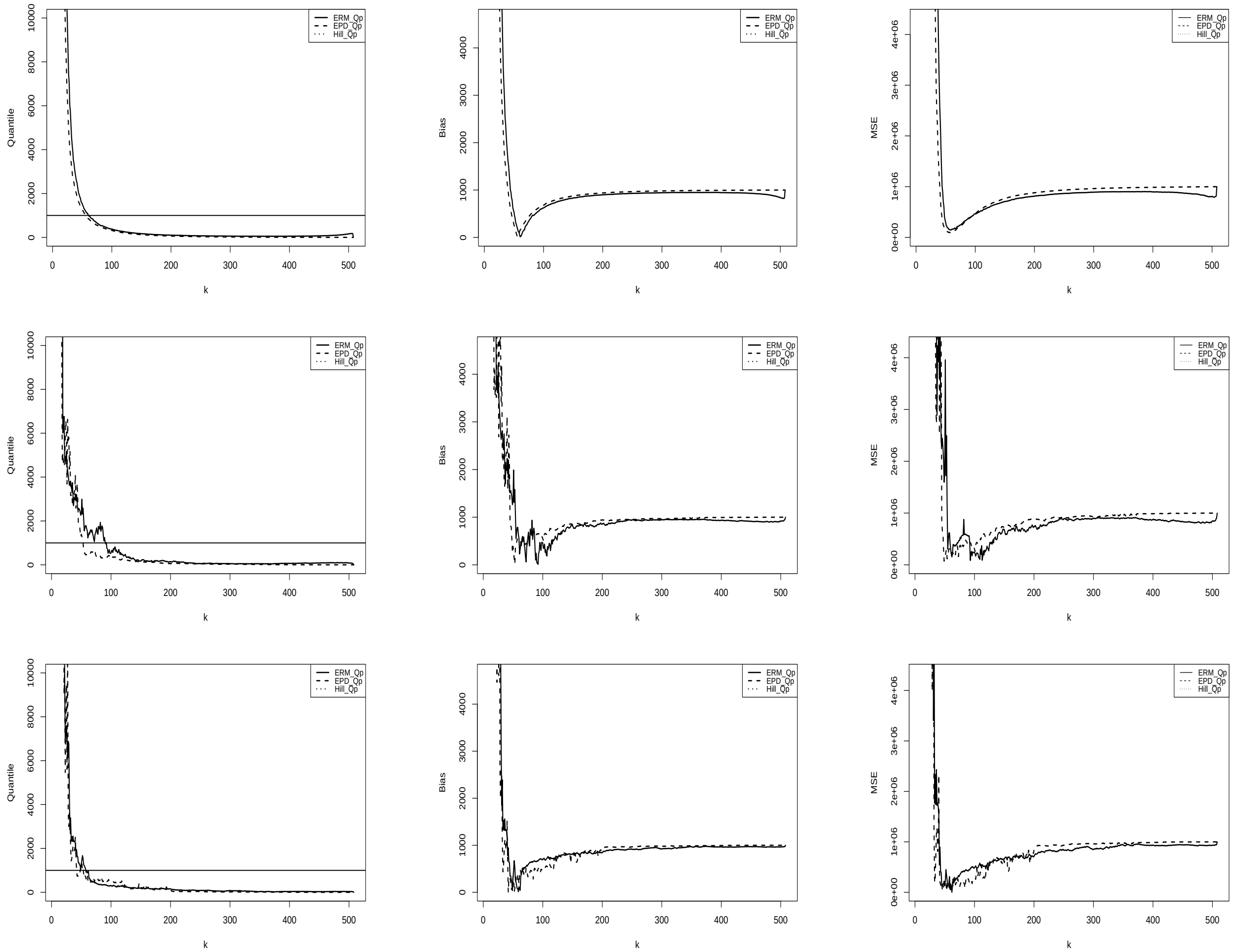

Figure 8: Fréchet distribution with $n = 500$, $p_n = 0.001$, $\varepsilon = 10\%$ and $\gamma = 1$. Topmost row: $\alpha = 0.1$; Middlemost row: $\alpha = 0.5$; and bottommost row: $\alpha = 1.0$. Leftmost column: average of $\hat{Q}(1 - p_n)$; middlemost column: Bias; and rightmost column: MSE.