



Article title: An improved bound for length of addition chains producing $2^n - 1$

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AN IMPROVED BOUND FOR LENGTH OF ADDITION CHAINS PRODUCING $2^n - 1$

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ABSTRACT. In this paper we prove that there exists an addition chain producing $2^n - 1$ of length $\delta(2^n - 1)$ satisfying the inequality

$$\delta(2^n - 1) \leq 2n - 1 - \left\lfloor \frac{n-1}{2^{\lfloor \frac{\log n}{\log 2} \rfloor}} \right\rfloor - \lfloor \frac{\log n}{\log 2} \rfloor + \iota(n)$$

where $\lfloor \cdot \rfloor$ denotes the floor function and $\iota(n)$ the shortest addition chain producing n .

1. Introduction

An addition chain producing $n \geq 3$, roughly speaking, is a sequence of numbers of the form $1, 2, s_3, s_4, \dots, s_{k-1}, s_k = n$ where each term is the sum of two earlier terms in the sequence, obtained by adding each sum generated to an earlier term in the sequence. The length of the chain is determined by the number of entries in the sequence excluding n . There are numerous addition chains that result in a fixed number n . The shortest or optimal addition chain produces n . However, given that there is currently no efficient method for getting the shortest addition yielding a given number, reducing an addition chain might be a difficult task. This makes addition chain theory a fascinating subject to study. Arnold Scholz conjectured the inequality by letting $\iota(n)$ denote the length of the shortest addition chain producing n .

Conjecture 1.1 (Scholz). The inequality holds

$$\iota(2^n - 1) \leq n - 1 + \iota(n).$$

It has been shown computationally that the conjecture holds for all $n \leq 5784688$ and in fact it is an equality for all $n \leq 64$ [2]. Alfred Brauer proved the scholz conjecture for the star addition chain, an addition chain where each term obtained by summing uses the immediately subsequent number in the chain. By denoting the shortest length of the star addition chain by $\iota^*(n)$, it is shown that (See,[1])

Theorem 1.1. *The inequality holds*

$$\iota^*(2^n - 1) \leq n - 1 + \iota^*(n).$$

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In this paper we study short addition chains producing numbers of the form $2^n - 1$ and the scholz conjecture. We adopt the method of **filling the potholes** to obtain an explicit improved upper bound for an addition chain producing $2^n - 1$.

2. Sub-addition chains

In this section we introduce the notion of sub-addition chains.

Definition 2.1. Let $n \geq 3$, then by the addition chain of length $k - 1$ producing n we mean the sequence

$$1, 2, \dots, s_{k-1}, s_k$$

where each term s_j ($j \geq 3$) in the sequence is the sum of two earlier terms, with the corresponding sequence of partition

$$2 = 1 + 1, \dots, s_{k-1} = a_{k-1} + r_{k-1}, s_k = a_k + r_k = n$$

with $a_{i+1} = a_i + r_i$ and $a_{i+1} = s_i$ for $2 \leq i \leq k$. We call the partition $a_i + r_i$ the i th **generator** of the chain for $2 \leq i \leq k$. We call a_i the **determiners** and r_i the **regulator** of the i th generator of the chain. We call the sequence (r_i) the regulators of the addition chain and (a_i) the determiners of the chain for $2 \leq i \leq k$.

Definition 2.2. Let the sequence $1, 2, \dots, s_{k-1}, s_k = n$ be an addition chain producing n with the corresponding sequence of partition

$$2 = 1 + 1, \dots, s_{k-1} = a_{k-1} + r_{k-1}, s_k = a_k + r_k = n.$$

Then we call the sub-sequence (s_{j_m}) for $1 \leq j \leq k$ and $1 \leq m \leq t \leq k$ a **sub-addition** chain of the addition chain producing n . We say it is **complete** sub-addition chain of the addition chain producing n if it contains exactly the first t terms of the addition chain. Otherwise we say it is an **incomplete** sub-addition chain.

2.1. Summary sketch and idea of proof. In this section we describe the method of **filling the potholes** which is employed to obtain our upper bound. We lay them down chronologically as follows.

- We first construct a complete sub-addition chain producing $2^n - 1$. For technical reasons which will become clear later, we stop the chain prematurely at 2^{n-1} .
- We extend this addition chain by a length of logarithm order.
- This extension has missing terms to qualify as addition chain producing $2^n - 1$. We fill in the missing terms thereby obtaining what one might refer to as spoof addition chain producing $2^n - 1$.
- Creating this spoof addition chain comes at a cost. The remaining step will be to cover the cost and render an account to obtain the upper bound.

3. Addition chains of numbers of special forms and Main result

In this section, we prove an explicit upper bound for the length of the shortest addition chain producing numbers of the form $2^n - 1$. We begin with the following important but fundamental result.

Lemma 3.1. *Let $\iota(n)$ denotes the shortest addition chain producing n . Then we have the inequality*

$$\lfloor \frac{\log n}{\log 2} \rfloor \leq \iota(n).$$

Theorem 3.2. *There exists an addition chain producing $2^n - 1$ of length $\delta(2^n - 1)$ satisfying the inequality*

$$\delta(2^n - 1) \leq 2n - 1 - \left\lfloor \frac{n-1}{2^{\lfloor \frac{\log n}{\log 2} \rfloor}} \right\rfloor - \lfloor \frac{\log n}{\log 2} \rfloor + \iota(n)$$

where $\lfloor \cdot \rfloor$ denotes the floor function.

Proof. First, let us construct the shortest addition chain producing 2^n as $1, 2, 2^2, \dots, 2^{n-1}, 2^n$ with corresponding sequence of partition

$$2 = 1 + 1, 2 + 2 = 2^2, 2^2 + 2^2 = 2^3 \dots, 2^{n-1} = 2^{n-2} + 2^{n-2}, 2^n = 2^{n-1} + 2^{n-1}$$

with $a_i = 2^{i-2} = r_i$ for $2 \leq i \leq n+1$, where a_i and r_i denotes the determiner and the regulator of the i th generator of the chain. Let us consider only the complete sub-addition chain

$$2 = 1 + 1, 2 + 2 = 2^2, 2^2 + 2^2 = 2^3 \dots, 2^{n-1} = 2^{n-2} + 2^{n-2}.$$

Next we extend this complete sub-addition chain by adjoining the sequence

$$2^{n-1} + 2^{\lfloor \frac{n-1}{2} \rfloor}, 2^{n-1} + 2^{\lfloor \frac{n-1}{2} \rfloor} + 2^{\lfloor \frac{n-1}{2^2} \rfloor} \dots, 2^{n-1} + 2^{\lfloor \frac{n-1}{2} \rfloor} + 2^{\lfloor \frac{n-1}{2^2} \rfloor} + \dots + 2^1.$$

We note that the adjoined sequence contributes at most

$$\lfloor \frac{\log n}{\log 2} \rfloor \leq \iota(n)$$

terms to the original complete sub-addition chain, where the upper bound follows by virtue of Lemma 3.1. Since the inequality holds

$$2^{n-1} + 2^{\lfloor \frac{n-1}{2} \rfloor} + 2^{\lfloor \frac{n-1}{2^2} \rfloor} + \dots + 2^1 < \sum_{i=1}^{n-1} 2^i = 2^n - 2$$

we insert terms into the sum

$$(3.1) \quad 2^{n-1} + 2^{\lfloor \frac{n-1}{2} \rfloor} + 2^{\lfloor \frac{n-1}{2^2} \rfloor} + \dots + 2^1$$

so that we have

$$\sum_{i=1}^{n-1} 2^i = 2^n - 2.$$

Let us now analyze the cost of filling in the missing terms of the underlying sum. We note that we have to insert $2^{n-2} + 2^{n-3} + \dots + 2^{\lfloor \frac{n-1}{2} \rfloor + 1}$ into (3.1) and this is comes at the cost of adjoining

$$n - 2 - \lfloor \frac{n-1}{2} \rfloor$$

terms to the term in (3.1). The last term of the adjoined sequence is given by

$$(3.2) \quad 2^{n-1} + (2^{n-2} + 2^{n-3} + \dots + 2^{\lfloor \frac{n-1}{2} \rfloor + 1}) + 2^{\lfloor \frac{n-1}{2} \rfloor} + 2^{\lfloor \frac{n-1}{2^2} \rfloor} + \dots + 2^1.$$

Again we have to insert $2^{\lfloor \frac{n-1}{2} \rfloor - 1} + \dots + 2^{\lfloor \frac{n-1}{2^2} \rfloor + 1}$ into (3.2) and this comes at the cost of adjoining

$$\lfloor \frac{n-1}{2} \rfloor - \lfloor \frac{n-1}{2^2} \rfloor - 1$$

terms to the term in (3.2). The last term of the adjoined sequence is given by

(3.3)

$$2^{n-1} + (2^{n-2} + 2^{n-3} + \dots + 2^{\lfloor \frac{n-1}{2} \rfloor + 1}) + 2^{\lfloor \frac{n-1}{2} \rfloor} + (2^{\lfloor \frac{n-1}{2} \rfloor - 1} + \dots + 2^{\lfloor \frac{n-1}{2^2} \rfloor + 1}) + 2^{\lfloor \frac{n-1}{2^2} \rfloor} + \dots + 2^1.$$

By iterating the process, it follows that we have to insert into the immediately previous term by inserting into (3.3) and this comes at the cost of adjoining

$$\lfloor \frac{n-1}{2^{k-1}} \rfloor - \lfloor \frac{n-1}{2^k} \rfloor - 1$$

terms to the term in (3.3) for $1 \leq k \leq \lfloor \frac{\log n}{\log 2} \rfloor$ since we filling in at most $\lfloor \frac{\log n}{\log 2} \rfloor$ blocks. It follows that the contribution of these new terms is at most

$$(3.4) \quad n - 1 - \left\lfloor \frac{n-1}{2^{\lfloor \frac{\log n}{\log 2} \rfloor}} \right\rfloor - \left\lfloor \frac{\log n}{\log 2} \right\rfloor$$

obtained by adding the numbers in the chain

$$n - 1 - \lfloor \frac{n-1}{2} \rfloor - 1$$

$$\lfloor \frac{n-1}{2} \rfloor - \lfloor \frac{n-1}{2^2} \rfloor - 1$$

⋮

⋮

$$\left\lfloor \frac{n-1}{2^{\lfloor \frac{\log n}{\log 2} \rfloor - 1}} \right\rfloor - \left\lfloor \frac{n-1}{2^{\lfloor \frac{\log n}{\log 2} \rfloor}} - 1 \right\rfloor.$$

By undertaking a quick book-keeping, it follows that the total number of terms in the constructed addition chain is at most

$$\delta(2^n - 1) \leq n + n - 1 - \left\lfloor \frac{n-1}{2^{\lfloor \frac{\log n}{\log 2} \rfloor}} \right\rfloor - \left\lfloor \frac{\log n}{\log 2} \right\rfloor + \iota(n)$$

thereby ending the construction. \square

Remark 3.3. We obtain a slightly weaker but much more explicit weaker version of Scholz's conjecture.

Corollary 3.1. Let $\iota(n)$ denotes the length of the shortest addition chain producing n . Then the inequality holds

$$\iota(2^n - 1) \leq 2n - 1 - \left\lfloor \frac{n-1}{2^{\lfloor \frac{\log n}{\log 2} \rfloor}} \right\rfloor - \left\lfloor \frac{\log n}{\log 2} \right\rfloor + \iota(n)$$

where $\lfloor \cdot \rfloor$ denotes the floor function.

$$\frac{1}{1}$$

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