IN PARTIAL SATISFACTION OF MEVSEM '17

PHYSICAL APPLICATIONS OF TOPOLOGICAL QUANTUM FIELD THEORY

Introduction to Chern-Simons Theory

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ABSTRACT: The 2 + 1 Yang-Mills theory allows for an interaction term called the Chern-Simons term. This topological term plays a useful role in understanding the field theoretic description of the excitation of the quantum hall system such as Anyons. While solving the non-Abelian Chern-simons theory is rather complicated, its knotty world allows for a framework for solving it. In the framework, the idea was to relate physical observables with the Jones polynomials. In this note, I will summarize the basic idea leading up to this framework.

Contents

0 Prologue

This note is a report in partial satisfaction of my study and seminar work on "*Physical Applications of Topological Quantum Field Theory*" at the University of Heidelberg. It is fair to declare forward that, while I have made efforts to carefully craft the basic ideas I have tried understood about a very fascinating topic within a very limited time, there are chances of omission or partial coverage of some concepts. I tried as much to cite important references where I consider it needful.

We shall study Witten's original paper on *"Quantum field theory and the Jones polynomial."* [1]. Elementary readers like me will agree that without some formal background understanding, this article belong to a class usually tagged "hard to read", more because it assumes many important advances, not only in physics but also in mathematics, as prerequisite. As such, I consider a short introduction about few background stories as useful and motivating towards our study. Following that, I will discuss the important elements of model description of anyons. The features introduced will play important roles throughout the remaining sections. I will follow similar structure of the lecture, but more explicit in this note. Our final destination is the evaluation of Wilson link invariant of the non-Abelian Chern-simon theory. The aim is to derive the rule for unknotting the Wilson link and compute the expectation value by relating physical observables to Jones polynomials which are knots invariant. As such, the roads to our final destination demands few important branches to transit towards an expected end.

1 Introduction

One of the ideas that revolutionized 20th century physics is Quantum field theory – a theoretical framework for modeling quantum mechanical system of elementary particles. Basically, the world is quantum, classicality is a certain limit of it and fields are nothing but part of reality. The emergent particles are excitation of fields. While in classical mechanics, particles are considered as distinguishable, new features arise on transition into Quantum theory. Among them is the fact that identical particles are indistinguishable. As a consequence, interchanging particles in a multiparticle state does not lead to a new configuration of the system, so that all probabilities is the same under such operation, i.e.

$$|\psi(\pi(\mathbf{r}_1, ..., \mathbf{r}_n))|^2 = |\psi(\mathbf{r}_1, ..., \mathbf{r}_n)|^2, \tag{1.1}$$

where π is the permutation operation of n particle co-ordinates. More precisely, the wavefunction is left invariant by the interchanging operation up to a certain phase. This understanding led to quantum statistical decription of particles, where interchanging operation in three or higher spatial dimension leads to an interpretation of wavefunction as symmetric or antisymmetric under exchange depending on whether it transforms as boson or fermion in the following sense:

$$\psi(\mathbf{r}_1, \mathbf{r}_2) \longrightarrow \eta\ \psi(\mathbf{r}_2, \mathbf{r}_1) \longrightarrow \eta^2\psi(\mathbf{r}_1, \mathbf{r}_2), \qquad \eta = \begin{cases} +1, & boson \\ -1, & fermion. \end{cases} \tag{1.2}$$

The study of planar physics is not a new thing in physical study. In particular, the 2D physics received some attention and rapid progress in the 70–90s towards understanding of possible particle statistics and their behaviour. On the first hand is an experimental physics leading to the discovery of fractional quantum Hall effect[2]. Way back in 1879, Edwin Hall had discovered the classical

Hall effect as a result of the motion of charged particles in a magnetic field[3]. An interesting fact about the very fractional quantum Hall effect is that, the hall conductivity

$$\sigma_{xy} = \frac{e^2}{2\pi\hbar}\nu, \quad \nu \in \mathbb{Q} \tag{1.3}$$

is surprisingly quantized. On the other hand is a theoretical curiousity leading to the prediction of quasiparticles and quasiholes whose statistics are different from those of bosons and fermions. The major stages include the following:

- **Laughlin Wavefunction:** This wavefunction was proposed as an ansatz to the lowest Landau levels at filling fraction $\nu = \frac{1}{m}$ for $m \in \mathbb{Z}$, with the excitations taking fractional charge $\frac{e}{m}$ [4].

- **Statistics:** In a system restricted to two spatial dimensions, some emergent phenomena could be observed [5–8], namely, emergent particles turns out to obey statistics interpolating between Fermi–Dirac and Bose–Einstein. These fractionally charged quasiparticle excitations of the Laughlin state are called *Abelian Anyons* or simply *Anyons*. The excitations emerge with fractional statistical angle $\theta = \frac{\pi}{m}$ as their wavefunctions acquire phase factor of $\eta = e^{i\theta}$ when identical quasiparticles undergo an exchange operation similar to (1.2). When exchange operation which transforms the system's quantum state is not commutative, *non-Abelian anyons* arise. Examples of non-Abelian quasiparticle statistics are found in system of Fibonacci anyons and Ising anyons.

- **Hierarchy states:** The idea that a more generic category of fractional quantum hall effect defined by infinite continued filling fraction

$$\nu = \frac{1}{m \pm \frac{1}{\tilde{m}_1 \pm \frac{1}{\tilde{m}_2 \pm \dots}}} \tag{1.4}$$

  was proposed and later validated quantitatively. [6, 9]

The field theoretic framework underlying this planar physics phenomena is known as *Chern–simons theory* (CS). It is a topological quantum field theory (TQFT) featuring topological invariant observables. The field theory enjoys general covariance since there is no a priori choice of metric on the manifold on which the theory is formulated. As we shall see furthermore, the anyonic phenomena does not emerge from this theory as a result of gauge invariance, but rather as a consequence of the theory being quasi-invariant, i.e. while the action is not fully gauge invariant, the observables and therefore the partition function are fully gauge invariant.

Remarkably, the nature of anyons has inspired some practical purposes such as in topological quantum computation. The very features of Chern-Simons invariants have lead to new developments in mathematics. One of our goals in this study is to understand how quantum Chern-Simons invariant is related to the Jones invariants of link which have found useful applications in knot theory.

2 Elements of Anyon Models

"Since interchange of two of these particles can give any phase, I will call them generically anyons." – **Frank Wilczek** in [8]

There is a general structure that all non–Abelian anyon models are required to feature. In this section, our goal is to discuss these basic features characterizing a model description of anyon. The features arise as a consequence of the following insights:

- Anyons can be created or annihilated pairwisely;
- Anyons can be fused to form composite anyons;
- Anyons has braiding rules.

These features will play roles in the model description of anyons by the planar Chern–Simons theory.

2.1 Particle Type

A non-trivial anyonic system generally consists of multiple types of anyon. Therefore, to define an anyon model, we must declare all the distinct type of anyon in the model. A given anyon model is expected to have:

- a finite set of elementary particle specie, say, $\{\phi_a, \phi_b, \phi_c, ...\}$. Each particle is locally distinguished by its label $a, b, c, ...$ which we think of as topological charges. These topological charges are conserved quantum numbers.
- trivial particle $\phi_{\mathbb{1}}$ which corresponds to a unique vacuum.
- antiparticle $\phi_{\bar{a}}$ such that $\phi_a \times \phi_{\bar{a}} = \phi_{\mathbb{1}} = \phi_{\bar{\mathbb{1}}}$. This implies that particles are only created in pair from the vacuum.

The simplest non-trivial anyon model is therefore spanned by the particle set

$$\Xi = \{\phi_{\mathbb{1}}, \phi_a, \phi_b, \phi_c, ...\}. \tag{2.1}$$

2.2 Fusion Rule

Given that anyons come in multiple, bound states could exist in principle. Even when no stable bound state exists, composite anyon can be formed by bringing two anyons close together. The fusion rule of Abelian anyon and the quantum number of the eventual composite particles is straightforward. There is only one possible fusion channel. For example, fusion of two given Abelian anyons with statistic $\frac{\alpha^2\pi}{m}$ and $\frac{\beta^2\pi}{m}$ is given as

$$\frac{\alpha^2\pi}{m} \times \frac{\beta^2\pi}{m} = \frac{(\alpha+\beta)^2\pi}{m}. \tag{2.2}$$

What is the statistical behaviour of composite non-Abelian anyon? Tensor product doesn't come in handy. The answer to this arise by recognizing that topological quantum numbers combine in non-unique manner. This means that there are different possible fusion channels. Fusion rule is given as

$$\phi_a \times \phi_b = \mathcal{N}^c_{ab}\phi_c, \quad \mathbb{Z} \ni \mathcal{N}^c_{ab} = \begin{cases} 1, & \textit{Abelian anyons} \\ \geq 2, & \textit{non} - \textit{Abelian anyons}. \end{cases} \tag{2.3}$$

Fusion is depicted in figure 1.The integer number $\mathcal{N}_{ab}^c$ is the number of distinguishable fusion channels. These fusion channels can be understood to be orthonormal basis states

$$\{|ab, c; \mu\rangle\,, \quad \mu = 1, 2, ..., \mathcal{N}_{ab}^c\} \tag{2.4}$$

of the fusion Hilbert space $\mathcal{V}_{ab}^c$ with the following properties:

- orthonormality: $\langle ab, c; \mu\, | ab, d; \nu\rangle = \delta_{cd}\delta_{\mu\nu}$
- completeness:
$$\sum_{\mu,c} |ab, d; \mu\rangle \langle ab, c; \mu| = I_{ab},$$
where I_{ab} is the trivial fusion channel.
- For non-Abelian anyons,
$$dim\left(\bigoplus_c \mathcal{V}_{ab}^c\right) = \sum_c \mathcal{N}_{ab}^c \geq 2$$
- fusion is associative, i.e.
$$\begin{cases} \phi_a \times \phi_b = \phi_b \times \phi_a \\ (\phi_a \times \phi_b) \times \phi_c = \phi_a \times (\phi_b \times \phi_c). \end{cases}$$
This basically insist that the order of tensor product is irrelevant, so that one order of fusing three anyons can be rotated into another in the fusion space by the Fusion matrix $\mathcal{F}_{abc}^d$ as described in figure 2.

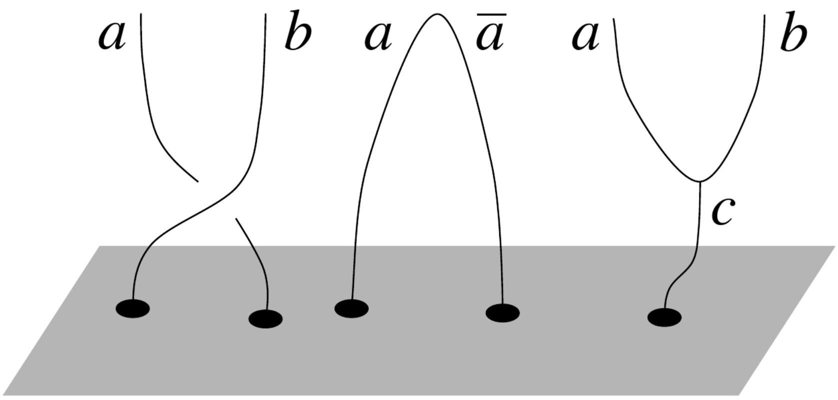

Figure 1. Worldlines of anyons. The first diagram features anyonic fundamental braid. Second diagram depicts pair-creation from the vacuum of a particle a and its antiparticle $\bar{a}$. The third describes the fusion of two particles a and b by two lines fusion into composite c

The simplest model of anyonic system is Fibonacci anyons[11] . It is an anyonic system having only two particle types: the vacuum $\phi_{\mathbb{1}}$ and the nontrivial particle type ϕ_τ. Fibonacci anyons are self-dual, i.e. particle τ is the same as its anti-particle. Fusion rules just go by

$$\begin{cases} \phi_\tau \times \phi_{\mathbb{1}} = \phi_{\mathbb{1}} \times \phi_\tau = \phi_\tau \\ \phi_\tau \times \phi_\tau = \phi_{\mathbb{1}} \oplus \phi_\tau, \end{cases}$$

implying that the dimension of fusion Hilbert space is 2.

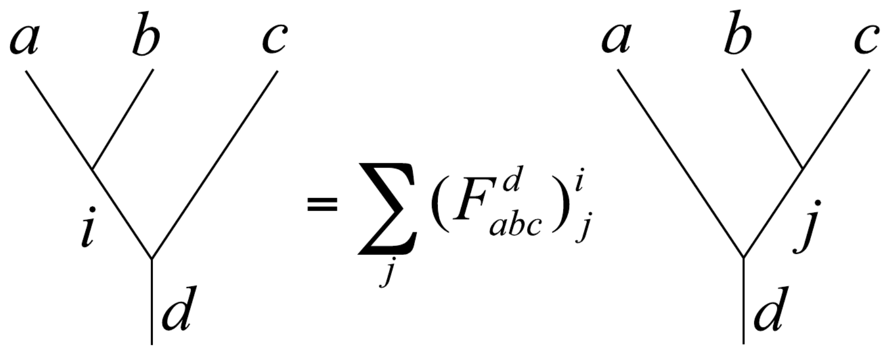

Figure 2. Order of fusion mapped by sufficient number of moves implemented by Fusion matrix $\mathcal{F}^d_{abc}$. Each order of fusion corresponds to a choice of basis with $\mathcal{F}$ matrix as a transformation between different bases. That i and j are different is consistent with having a fixed composite fusion outcome d.

2.3 Artin's braid group

In the introduction, we recognised in (1.1) that under the exchange of particles in $3+1$ spacetime dimensions, wavefunction describing a system of identical particles acquires $\pm$ phase and therefore exhibits two possible types of symmetry depending on whether it is a boson or a fermion. In path integral interpretation, twice operation of particle exchange is equivalent to a process in which one of the particles is taken along a trajectory that wrap around the other. Different trajectories belong to topological class corresponding to the elements of the permutation group S_N specifying how the initial positions are permuted into the final positions. These trajectories in three spatial dimensions can be continuously deformed into straight in time direction. As such, the wave function is left invariant under exchange operation.

We further learnt that new statistical behaviour emerges in $2+1$ spacetime dimensions, and anyons can exist in principle. The precise statement is that, under twice operation of particle exchange, η^2 isn't necessarily equals 1, so that the wavefunction need not return to the initial state, i.e. it acquires a non–trivial phase such that

$$\psi(\mathbf{r}_1, \mathbf{r}_2) \longrightarrow e^{i\theta}\psi(\mathbf{r}_2, \mathbf{r}_1) \longrightarrow e^{i2\theta}\psi(\mathbf{r}_1, \mathbf{r}_2), \tag{2.5}$$

and the special case of boson and fermions are realized for $\theta = 0, \pi$ respectively. Particles with statistical angle θ interpolating between 0 and π are called anyons. In general case of n-particle state $\psi(\mathbf{r}_1, ..., \mathbf{r}_n)$, the topological classes of trajectories which take these particles from initial positions $(t_i, \mathbf{r}_1, ..., \mathbf{r}_n)$ to final position $(t_f, \mathbf{r}_1, ..., \mathbf{r}_n)$ is in one-to-one correspondence with the elements of the braid group B_n. In figure 1 is a description of anyon worldlines originating from a plane with a characteristic fundamental braid.

Braid is an everyday concept but the formal definition of an n-braid as a topological object was given by Artin[10]. Consider two parallel frames with each frame as a plane in euclidean 3-space. Let $\mathbf{p}_i$ $(i = 1, ..., n)$ be distinct mark ponts on the lower frame Σ_L and denote their orthogonal projections onto the upper frame Σ_U as $\mathbf{q}_i$. Each $\mathbf{p}_i$ can be joined with $\mathbf{q}_j$ by means of strings which intersect any parrallel plane between Σ_L and Σ_U exactly once.

A braid is an intertwining of some number of strings $f_i : I = [0, 1] \longmapsto \Sigma$ such that:

1. $f_i(0) = \mathbf{p}_i$,
2. $f_i(1) = \mathbf{p}_{\pi(i)} = \mathbf{q}_i$ for some for some permutation π,
3. Given that $i \neq j$, then $f_i(t) \neq f_j(t)$ for each $t \in I$.

The braid group B_n therefore consists of the set of all braids with n strings under multiplication operation given by glueing. The braid group is furnished in a presentation provided in the following theorem:

Theorem (Artin[10]): Let $\mathcal{B}_n$ be a group generated by $\sigma_1, \sigma_2, ..., \sigma_{n-1}$ modulo the following relations:

1. $\sigma_i\sigma_j = \sigma_j\sigma_i$ for $|i - j| \geq 2$.
2. $\sigma_i\sigma_{i+1}\sigma_i = \sigma_{i+1}\sigma_i\sigma_{i+1}$ for $i = 1, ..., n - 2$,

There is an isomorphism $\rho : \mathcal{B}_n \longmapsto B_n$ determined by σ_i.

The elementary braids are those generated by σ_i with just one crossing. The group $\mathcal{B}_n$ is different from permutation group in that $\sigma_i^2 \neq 1$. As a consequence, while the permutation group is finite with $|S_N| = N!$, the braid group is infinite. The two relations in the Artin's theorem are better revealed in figure 3 which notes that one composition can be smoothly deformed into the other. The first relation implies that the order of composition is important because the braid group is non-Abelian, e.g. $\sigma_1\sigma_2 \neq \sigma_2\sigma_1$. The second relation is nothing but Yang–Baxter equation characterizing quantum integrable system. The Yang–Baxter equation is a statement that the two possible paths that three strings can take are different but equivalent.

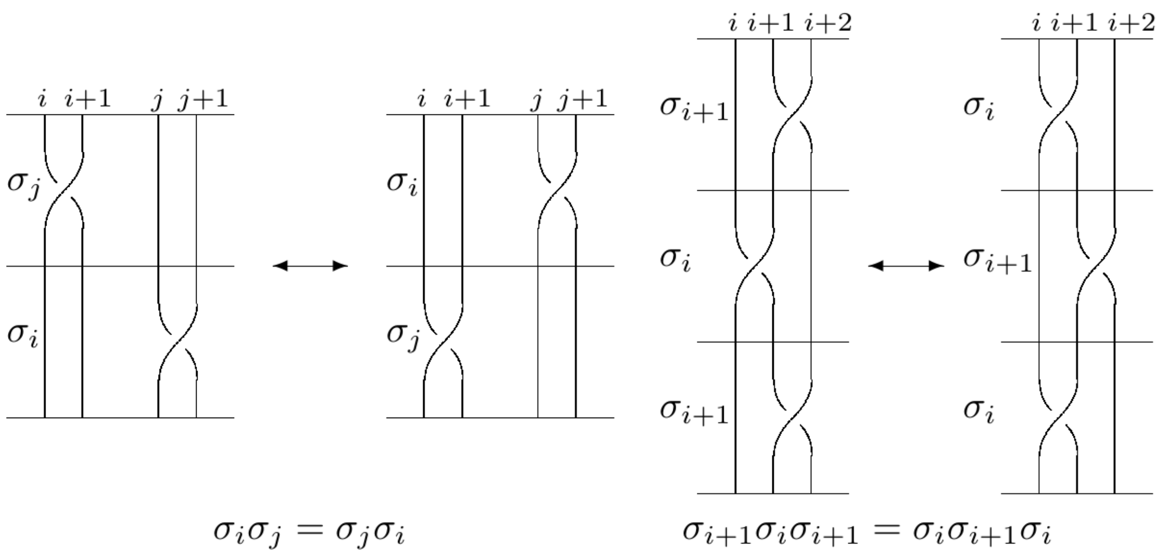

Figure 3. Relations in Artin's braiding.

3 Chern-Simons Theory

We have recognized Quantum field theory as a framework of model description of elementary particles. As we shall now demonstrate, the low energy behaviour of the fractional quantum Hall effect can be described by a certain planar Yang–Mill theory in the presence of Chern-Simons(CS) term [12]. In the following, we shall use both classical and quantum mechanical understanding to demonstrate that the source of Abelian and non-Abelian Chern-Simons theories have relevant anyonic features.

3.1 Classical CS Invariant

Let Σ be a surface embedded in 3D Euclidean space with a compact boundary $\partial\Sigma$. The Riemannian metric determines a measure $d\mu_\Sigma$ on Σ and defines the curvature of Σ as a function $K : \Sigma \to \mathbb{R}$. The curvature is intrinsic in that it depends only on the induced metric on Σ, and not on the embedding of Σ into space. The generalized Gauss-Bonnet formula is

$$\int_\Sigma K \, d\mu_\Sigma + \int_{\partial\Sigma} \kappa_{\partial\Sigma} \, d\mu_{\partial\Sigma} = 2\pi\chi(\Sigma) \tag{3.1}$$

where $\chi(\Sigma)$ is the Euler characteristic of Σ and κ is the total geodesic curvature of the boundary. The classical CS invariant[13] is a generalization of the total geodesic curvature.

Let G be a finite Lie group and t be its Lie algebra. We fix a certain closed oriented 3-manifold $\mathcal{M}$ with m-form $\Omega^m_{\mathcal{M}}(t)$ living on it. A connection $\Theta = \Theta^a_\mu t_a dx^\mu$ on the G-bundle over $\mathcal{M}$ is a skew-Hermitian matrix of 1-forms having trace zero. The curvature of Θ is a t-valued 2-form $\Omega^2(\Theta) = d\Theta + \frac{1}{2}[\Theta \wedge \Theta]$ satisfying the Bianchi identity

$$d\Omega + [\Theta \wedge \Omega] = 0.$$

The Chern-Simons 3-form is given as

$$\Omega^3(\Theta) = \Theta \wedge d\Theta + \frac{2}{3}[\Theta \wedge \Theta \wedge \Theta], \tag{3.2}$$

and the corresponding classical CS invariant is

$$S_{CS}(\Theta) = \frac{k}{4\pi}\int_{\mathcal{M}} \mathrm{Tr}\left(\Theta \wedge d\Theta + \frac{2}{3}[\Theta \wedge \Theta \wedge \Theta]\right), \tag{3.3}$$

with k being the level of the theory. CS field theories exist for all odd and higher dimensions. The generalized CS form in $2n+1$ dimensions reads as

$$\Omega^{2n+1}(\Theta) = \Theta \wedge (d\Theta)^n + \alpha_1\Theta^3 \wedge (d\Theta)^{n-1} + \ldots.. + \alpha_n\Theta^{2n+1} \tag{3.4}$$

and $\alpha_i \in \mathbb{Q}$ is fixed.

3.2 CS Model of Anyons

Let a_μ be an emergent gauge fields of a certain Yang-Mill theory. In QFT, all physical information are encoded in the n–point correlation functions which can be extracted from functional differential of the partition function

$$Z[a_\mu] = \int \mathcal{D}\Psi e^{iS[\Psi, a_\mu]} = e^{iS_{eff}[a_\mu]}, \tag{3.5}$$

where Ψ represent all dynamical fields which are integrated out towards arriving at the effective field action S_{eff}. The effective action describes some physics, accurate at the very low energy. Our first goal will be to identify the Chern-Simons action which describes the relevant physics of interest.

Abelian Chern-Simons theory emerge as a possible generalization of 3+1 electromagnetic $U(1)$ gauge theory to 2+1 dimensions. Locally in four spacetime dimensions, gauge connection a_μ is described by massless degrees of freedom, whose dynamics are encoded in the Maxwell's equations obtained by extremizing the action

$$S_{maxwell}[a_\mu] = -\frac{1}{4e^2}\int_{\mathcal{M}} d^4x \; f_{\mu\nu}f^{\mu\nu}, \tag{3.6}$$

where $f_{\mu\nu} = \partial_\mu a_\nu - \partial_\nu a_\mu$ is the curvature known as the electromagnetic field strength. We adopt a manifold $\mathcal{M}$ as a space on which the theory is formulated. We assume $\mathcal{M}$ has trivial topology with boundary $\partial\mathcal{M}$ arbitrarily far from particle worldlines, so that all field vanish on the boundary. The action is the most general Lorentz invariant action compatible with gauge transformations $a_\mu \to a_\mu + \partial_\mu\alpha$. The massless gauge field is in turn a consequence of gauge invariance which forbids some forms of interaction including a mass term. However in three spacetime dimensions, new interaction terms is allowed in the full action

$$S = S_{maxwell} + S_{CS},$$

which can potentially change the dynamics of the system. This term establishes the Abelian Chern-Simons action

$$S_{CS}[a] = \frac{k}{4\pi}\int_{\mathcal{M}} d^3x \;\; \epsilon^{\mu\nu\rho}\, a_\mu\partial_\nu a_\rho + (......) \tag{3.7}$$

where (......) may include some higher derivative terms. Before we explain why this action encodes an effective field theory describing the quantum hall system, it might be useful to generalize the Chern-Simons action into non-Abelian just as in the usual QFT formalism. We recognize the relevant Lie gauge group $G = SU(N)$ with $a_\mu = a^c_\mu t^c$ considered as Lie(G)-valued gauge potential. The killing form $\kappa^{ab} = tr\left(t^a t^b\right) = \frac{1}{2}\delta^{ab}$ can be used to raise and lower indices of the Lie-algebra. The curvature associated with the connection is

$$f_{\mu\nu} = \partial_\mu a_\nu - \partial_\nu a_\mu - i[a_\mu, a_\nu].$$

The full action in 2+1 dimension is therefore nothing but a quantum Yang-Mills theory, with an action consisting of the Chern-Simons term that has additional 3-point gauge structure, i.e.

$$S = S_{YM} + S_{CS}$$

with

$$S_{CS}[a] = \frac{k}{4\pi}\int_{\mathcal{M}} d^3x \;\; \epsilon^{\mu\nu\rho} \;\mathrm{Tr}\left(a_\mu\partial_\nu a_\rho - \frac{2i}{3}a_\mu a_\nu a_\rho\right). \tag{3.8}$$

This is the action of level k non-Abelian Chern-Simons theory with gauge group G_k. It takes the same structure as the Chern-Simons invariant (3.3). The most important gauge group for the purpose of our study will be $SU(2)_k$.

So, what are the features of Chern-simons theory that makes it a good effective field theory of the quantum hall system? In the following, we shall highlight both the features and some physics associated with the hall system.

- The Chern-Simons action has no metric notion, instead, the Levi-Civita $\epsilon^{\mu\nu\rho}$ plays the role of raising and lowering. As such, the CS action is invariant under all diffeomorphism. The consequence of this is that, all n-point correlation functions are independent of the metric of spacetime, i.e. $\frac{\delta}{\delta g_{\mu\nu}}\langle \mathcal{O}_1(x_1)....\mathcal{O}_n(x_n)\rangle = 0$. This make the correlation functions topological invariants and Chern-Simons theory a good example of topological quantum field theory.

- On dimensional ground, the Chern-Simons term dominates over the Yang-Mill and (......) term in 2+1 dimension as it contains only one derivative relative to higher derivatives in the other terms. With Yang-Mill and other terms suppressed, the relevant long distance physics of the hall system is completely described by the Chern-Simons effective local Lagrangian, provided that $k \neq 0$.

- The Chern-simons action is invariant under rotation but violate parity $(x \to -x)$ and time reversal $(t \to -t)$ invariance. This is exactly the feature exhibited by quantum hall system where parity and time reversal are broken due to the external or background magnetic field as particles are restricted to a plane.

- Conservation of current plus the fact that vector theory demands the gauge field couples with the current implies that the current is the curl of the vector potential. Extremizing the CS action (3.7), we have the current as

$$J^\mu = \frac{\delta S_{CS}[a]}{\delta a_\mu} = \frac{k}{2\pi}\epsilon^{\mu\nu\rho}\,\partial_\nu a_\rho \quad\Longrightarrow\quad J_i = -\frac{k}{2\pi}\epsilon_{ij}E^j.$$

  This means that, Chern-Simons action describes a system with conductivity

$$\sigma_{xy} = \frac{k}{2\pi}. \tag{3.9}$$

  We shall soon understand that, indeed, this is the hall conductivity of the very quantum hall effect if we identify the Chern-Simons level k with ν landau level of the quantum hall system.

- Under a gauge transformation

$$a_\mu \longrightarrow a_\mu^u = u^{-1}a_\mu u + iu^{-1}\partial_\mu u \tag{3.10}$$

  with $u(x) \in G$, the Chern-Simons action (3.8) transforms as

$$\begin{aligned} S_{CS}[a] \longrightarrow S_{CS}[a^u] = S_{CS}[a] &+ \frac{k}{4\pi}\int_{\mathcal{M}} d^3x\;\; \epsilon^{\mu\nu\rho}\,\partial_\mu\,\mathrm{Tr}\left(\partial_\nu u u^{-1}a_\rho\right) \\ &+ \frac{k}{12\pi}\int_{\mathcal{M}} d^3x\;\; \epsilon^{\mu\nu\rho}\;\mathrm{Tr}\left(u^{-1}\partial_\mu u\, u^{-1}\partial_\nu u\, u^{-1}\partial_\rho u\right). \end{aligned} \tag{3.11}$$

  This means that the action changes by a total derivative and an additional term

$$\mathbb{Z} \ni \varsigma(u) = \frac{1}{24\pi^2}\int_{\mathcal{M}} d^3x\;\; \epsilon^{\mu\nu\rho}\;\mathrm{Tr}\left(u^{-1}\partial_\mu u\, u^{-1}\partial_\nu u\, u^{-1}\partial_\rho u\right) \tag{3.12}$$

  Recognizing the total derivative in (3.11), the purely surface integral vanishes on the boundary. This follows because we have demanded all field to vanish on the boundary. However, the term involving $\varsigma(u)$ does not necessarily vanish if we demand $u(x) \to \mathbb{1}$ at infinity over the entire argument. Since the gauge transformation $u(x)$ is defined throughout $\mathbb{R}^3$ and its argument is equivalent to $\mathbb{S}^3$, the local gauge transformation is a map induced by a change of coordinates, i.e. $u : \mathbb{S}^3 \to G$, and $\varsigma(u)$ measures the number of time the mapping winds around the spacetime. This therefore implies that the Chern-Simons action transformaming as

$$S_{CS}[a] \longrightarrow S_{CS}[a^u] = S_{CS}[a] + 2\pi k\, \varsigma(u)$$

is not invariant under gauge transformation. At this point, it appears gauge invariance is an obstacle towards having CS as a good model description of anyon, however, this violation is precisely all that is left to produce exact same scenario of the hall conductivity. In QFT, the object that must actually demand full compliance of gauge invariance isn't the action itself, but the very partition function (3.5) that encodes all information of the theory. Indeed, if we demand the Chern-Simons level k to be an integer, so that on reinstating the natural units $e = 1 = \hbar$, we can achieve

$$\frac{\hbar k}{e^2} \in \mathbb{Z}, \tag{3.13}$$

then the partition function $Z[a_\mu]$ is always gauge invariant. In fact, by associating $\frac{\hbar k}{e^2}$ with the filling fraction ν in (3.9) i.e. $\nu = \frac{\hbar k}{e^2}$, we precisely reproduce the quantized hall conductivity $\sigma_{xy} = \frac{e^2}{2\pi\hbar}\,\nu$ in (1.3). This is an important result arising from CS theory being quasi-invariant.

- It is worth noting that the hierarchy state filling fraction in (1.4) can be constructed from this formalism by introducing N emergent gauge fields a^i_μ with $i = 1, ..., N$. The full effective theory with quasiparticle excitations is given by[14]

$$\mathcal{L} = \frac{1}{4\pi} K_{ij} \epsilon^{\mu\nu\rho}\, a^i_\mu \partial_\nu a^j_\rho + \frac{1}{2\pi} t_i \epsilon^{\mu\nu\rho} A_\mu \partial_\nu a^i_\rho \tag{3.14}$$

  where the theory is completely specified by the CS couplings K_{ij} and the charge vector t_i. The K-matrix associated with the hierarchy in (1.4) is given as

$$K_{ij} = \begin{bmatrix} m & -1 & 0 & \dots \\ -1 & \widetilde{m}_1 & -1 & \\ 0 & -1 & \widetilde{m}_2 & \\ \vdots & & & \ddots \end{bmatrix} \tag{3.15}$$

- A quick check into the field equation of the full theory shows that the gauge fields have acquired mass which decay exponentially according as e^{-mr}, thereby rendering them as short ranged. This shows that, just like in the Aharonov-Bohm effect, the field can take large values away from the sources.

By now, we have understood how Chern-Simons theory makes a good model description of certain quasiparticles similar to those arising from the quantum hall system. Our next goal is to compute the expectation values of CS observables and then study how it features the anyonic properties of the source.

4 Wilson Loops and Anyonic Statistics

Solving a QFT is up to determination of the n-point correlation functions. For Chern-Simons theory, we would like to evaluate the topological invariants of the theory. These are essentially the expectation value of a certain observables $\mathcal{O}_i$ on the manifold $\mathcal{M}$, i.e.

$$\langle \mathcal{O}_1(x_1)....\mathcal{O}_n(x_n)\rangle = \frac{1}{Z}\int_{\mathcal{M}} \mathcal{D}a\, e^{iS_{CS}[a]}\mathcal{O}_1(x_1)....\mathcal{O}_n(x_n). \tag{4.1}$$

In this evaluation, anyonic statistics will depends on the evolution of wavefunction under the braiding operation of worldlines of anyons. We already saw braiding as a topological property of anyons. We will now study how this arise from CS theoretic description.

4.1 Abelian and non-Abelian

Our first goal is to identify an appropriate observable on the manifold. Define the Chern-Simons source as a set of particles on worldline in 2+1 spacetime dimensions. We would like to find a physical description for space of trajectory of charged particles living inside a Manifold $\mathcal{M}$ with boundary $\partial\mathcal{M}$ far away from worldlines, so that fields vanish on the boundary. Lets assign particle a spin j in the representation of CS gauge group G_k. We demand that any appropriate observable respect gauge invariance as the symmetry of the theory. There is a certain set of gauge invariant functional integral that satisfy this purpose. It is called the Wilson loop.

Starting with the Abelian. Given a 1-form a_μ as the gauge connection, an important concept is holonomy $H_{C,j}$, defined as the parallel transport along a closed curve C:

$$H_{C,j}[a] = \mathcal{P}\exp\left(iq\oint_C a_\mu dx^\mu\right), \tag{4.2}$$

where $\mathcal{P}$ is the path-ordering operator and q is the charge associated with the particle source with label j. Holonomies are important concept because any gauge invariant quantity involving the connection a_μ is a traces of holonomies. This gauge invariant object is the Abelian Wilson loop

$$W_{C,j}[a] = \mathrm{Tr}\left(\mathcal{P}\exp iq\oint_C a_\mu dx^\mu\right). \tag{4.3}$$

A simple interpretation will be to consider a source of quasiparticle-quasihole pair of type j, created from the vacuum, evolves along Wilson loop and then fused back to the vacuum. This process is described in figure 4. We shall soon see that the Chern-Simons level k restricts the allowed set of spin representation j propagating in the loop.

Loosely speaking, a collection of disjoint loops ($C_i, i = 1, ...N$) is called link (L) , so that we can define the product of Wilson loops as

$$W_{L,R}[a] = \mathrm{Tr}\left(\mathcal{P}\prod_{i=1}^{N}\exp iq_i\oint_{C_i} a_\mu dx^\mu\right). \tag{4.4}$$

We can generalize the Wilson loop into non-Abelian by recognising that the gauge connection takes its value from the Lie algebra spanned by t^c. We can set the charge as $q = 2j$ and define $t^c_{(j)} = 2j\, t^c$. The the non-Abelian Wilson loops in the representation R of the gauge group G_k is defined as

$$W_{L,R}[a] = \mathrm{Tr}_R\left(\mathcal{P}\prod_{i=1}^{N}\exp i\oint_{C_i} t^c_{(j)}a^c_\mu dx^\mu\right). \tag{4.5}$$

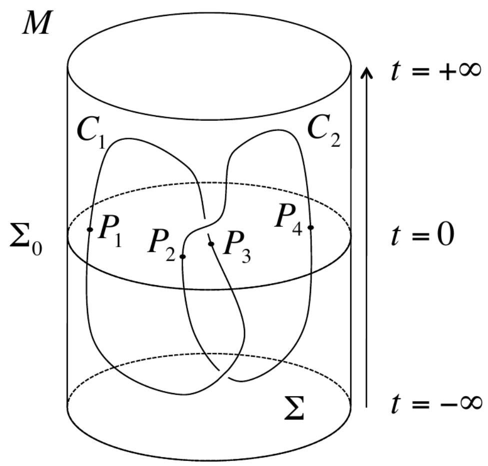

Figure 4. Worldlines of particles in a 3D Euclidean space $\mathcal{M}$. Wilson lines are defined by loops C_1 and C_2. The two loops together form a link. The link intersect disk Σ_0 at points P_i. The diagram depicts four particle sources pairwisely created from the vacuum; the source braided and then fused back to the vacuum

4.2 Expectation Value

I. Abelian CS

As we have noted in (4.1), the expectation value of the Wilson loop is given as

$$\langle W_{L,j} \rangle = \frac{\int_{\mathcal{M}} \mathcal{D}a \, W_{L,j} \, e^{iS_{CS}[a]}}{\int_{\mathcal{M}} \mathcal{D}a \, e^{iS_{CS}[a]}}. \tag{4.6}$$

Consider an Abelian Chern-Simons theory (3.7) and a link of disjoint loops C_i associated with charges q_i living in the worldvolume $\mathcal{M}$. Figure 4 depicts the case of two loops link. We are left with a process in a link configuration such that, particles are created pairwisely from the vacuum, propagate through the worldlines C_i and then fuse back to the vacuum. It is not hard to show in the case of Abelian CS, that the expectation value of the Wilson loops is given by[15]

$$\langle W_{L,j} \rangle = exp\left(\frac{i2\pi}{k} \sum_{i,j} q_i q_j \, \Phi(C_i, C_j) \right), \tag{4.7}$$

with Gauss linking number

$$\mathbb{Z} \ni \Phi(C_i, C_j) = \frac{1}{4\pi} \oint_{C_i} dx^\mu \oint_{C_j} dy^\nu \, \epsilon^{\mu\nu\rho} \, \frac{(x-y)^\rho}{|x-y|^3}, \quad for \;\; i \neq j. \tag{4.8}$$

$\Phi(C_i, C_j)$ and $\langle W_{L,j} \rangle$ are topological invariants (of link. $\Phi(C_i, C_j)$ measures the number of times one loop C_i winds around another loop C_j. It may be positive or negative depending on the orientation of the loops involved.

Comments:

1. The Gauss linking number is divergent for $i = j$. This can be resolved by a framing prescription that displace one of the loops, say C_i, slightly in a given direction to a loop C_i'. As such, a ribbon structure with boundaries C_i and C_i' is formed. This results in a well defined Gauss

linking number.

2. To display the anyonic nature of the source, we consider the trivial loops L_0 corresponding to disentangled loops. We note that an entagled loop corresponds to a braided sources characterizing anyons. Consider two loops C_1 and C_2 winding one another once according as $\Phi(C_1, C_2) = \pm 1$ so that their trivial link corresponds to two disentangled loops. Then braiding operation according to the expectation value (4.7) is given as

$$\langle W_{L,j} \rangle = exp\left(i\frac{4q^2}{k}\right)\langle W_{L_0,j} \rangle. \tag{4.9}$$

   Indeed, the braiding operation left a phase. This implies that the Chern-Simons sources have anyonic statistics.

3. The generalization of the expectation value to non-Abelian CS theory is complicated by large class of links having the same Gauss linking number $\Phi(C_i, C_j)$. We need a different algorithm to resolve this situation. This algorithm aim to evaluate a certain knot invariant called the Jones polynomials. However for now, consider two sources of a non-Abelian $SU(2)_k$ CS gauge group characterized by a label j. Fusion of j_1 and j_2 is dictated by vector composition

$$j_1 \otimes j_2 = \sum_{|j_1 - j_2|}^{\lambda(j_1,j_2)} j. \tag{4.10}$$

   Are all j allowed by fusion rules in CS theory? The answer is no as not all j gives distinct source . Indeed, expectation value of the non-Abelian Wilson loops for $j_{max} > \frac{k}{2}$ produce the same result as those of the representation with $0 \leq j \leq \frac{k}{2}$. Therefore, there are only $k+1$ allowed representations: $0, \frac{1}{2}, ..., \frac{k}{2}$ – with

$$\lambda(j_1, j_2) = \begin{cases} j_1 + j_2, & j_1 + j_2 \leq \frac{k}{2} \\ k - j_1 - j_2, & j_1 + j_2 > \frac{k}{2}. \end{cases} \tag{4.11}$$

II. Non-Abelian CS

We noted earlier that the generalization of the expectation value to non-Abelian CS theory is not straightforward. The complication arises due to a large class of links having the same Gauss linking number $\Phi(C_i, C_j)$. We state forward at this point that, we are yet to have enough technique to confront the problem. However, we shall lay down the formal approach towards the computation of the partition function. Here, we shall sketch out Witten's ideas. This idea will be used later to resolve through the problem at hand.

Witten's Procedure

Consider the three manifold M(picture it as S^3). Inside this manifold is our theory, i.e. Wilson loop $W_{C,j}$ is living therein as shown in figure 5.

- Cut the 3-manifold $\mathcal{M}$ along along a Riemann surface Σ, into two 3-manifolds $\mathcal{M}_1$ and $\mathcal{M}_2$. Obviously, $\mathcal{M}_1$ and $\mathcal{M}_2$ contain boundary surfaces, say Σ_1 and Σ_2, so that after cutting, the boundary of the pieces become $\partial\mathcal{M}_1 = \Sigma_1 \cup \Sigma$ and $\partial\mathcal{M}_2 = \Sigma_2 \cup \Sigma^*$. Σ^* is essentially Σ with opposite orientation. The surface of the cut would appear like $\Sigma \times \mathbb{R}$, where Σ is the spatial part and $\mathbb{R}$ is the time direction in a similar way to figure 4.

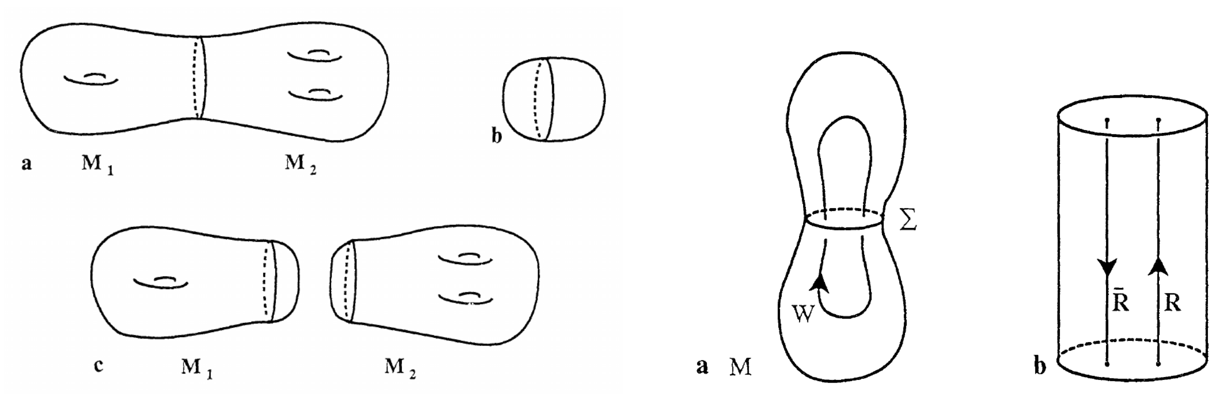

Figure 5. Witten's Quantization procedure. **Left:** (a) depicts the cutting procedure of a general manifold $\mathcal{M}$ into $\mathcal{M}_1$ and $\mathcal{M}_2$. One can do similar cutting procedure for a three sphere S^3 in (b). **Right:** (a) shows the inclusion of Wilson loop W and the cutting of the three manifold $\mathcal{M}$. Wilson loop carrying representation R of the gauge group pierce through Riemann surface Σ and leaves marked points on it. (b) shows that, near the cut, $\mathcal{M}$ looks locally like $\Sigma \times \mathbb{R}$.

- Canonically quantize CS theory on Σ and construct the physical Hilbert space $\mathcal{H}_\Sigma$. A path integral over $\mathcal{M}_1$ is a quantum state $|\psi_{\mathcal{M}_1}\rangle \in \mathcal{H}_\Sigma$. Since the surface Σ^* of $\partial\mathcal{M}_2$ has opposite orientation of Σ, the physical Hilbert space on $\partial\mathcal{M}_2$ is just a dual space $\mathcal{H}^*_\Sigma$ and the path integral over it is denoted as $\langle\psi_{\mathcal{M}_2}| \in \mathcal{H}^*_\Sigma$.

- **Heegaard splitting and Gluing:** Let Every $\mathcal{M}$, $\mathcal{M}_1$, and $\mathcal{M}_2$ be compact, closed, connected, orientable 3-manifolds and $\partial\mathcal{M}_1 = \partial\mathcal{M}_2 = \Sigma = \mathcal{M}_1 \cap \mathcal{M}_2$ be a Riemann surface. $\mathcal{M}$ admits Heegaard splitting $(\Sigma, \mathcal{M}_1, \mathcal{M}_2)$ such that, given an orientation reversing homeomorphism $f : \Sigma \to \Sigma$, there is a gluing procedure leading to $\mathcal{M} = \mathcal{M}_1 \cup_f \mathcal{M}_2$[17].

- **Atiyah's Axioms:** A TQFT is a functor $\mathcal{Z} : nCob \to \mathcal{H}$ with respect to orientation preserving diffeomorphisms of Σ and $\mathcal{M}$. This statement basically means that, we assign a Hilbert spaces $\mathcal{H}_\Sigma$ and $\mathcal{H}_\mathcal{M}$ to Rieman surface Σ and 3-manifolds $\mathcal{M}$ respectively. Functor $\mathcal{Z}$ is involutory, i.e. $\mathcal{Z}(\Sigma^*) = \mathcal{Z}(\Sigma)^*$ and multiplicative, i.e. $\mathcal{Z}(\Sigma_1 \cup \Sigma_2) = \mathcal{Z}(\Sigma_1) \otimes \mathcal{Z}(\Sigma_2)$ [18].

- The homeomorphism $f : \Sigma \to \Sigma$ induces a homeomorphism $U_f : \mathcal{H}_\Sigma \to \mathcal{H}^*_\Sigma$. U_f is a unitary operator so that we can compute the partition function $Z(\mathcal{M}) = \langle\psi_{\mathcal{M}_2}| U_f |\psi_{\mathcal{M}_1}\rangle$ on $\mathcal{M}$.

- Now insert the Wilson loops in $\mathcal{M}$. On Σ is left mark points or punctures P_i with each assigned a representation R_i of the gauge group G_k. This therefore allows us to quantize the theory on $\Sigma \times \mathbb{R}$, and the theory is solved on evaluating the expectation value

$$\langle W_{C_1,R_1} ... W_{C_n,R_n}\rangle = \frac{1}{Z}\int_{\mathcal{M}} \mathcal{D}a\, e^{iS_{CS}[a]} W_{C_1,R_1} ... W_{C_n,R_n}. \tag{4.12}$$

5 The Jones Polynomial

In this section, our goal is to get accustomed to the machinery we shall use in evaluating the expectation value of the non-Abelian Wilson loops. While the concept associated with the topic in this section is broad as an area of interest, all we need for our purpose is a mathematical concept called the Jones polynomial. Our knot excursion shall therefore be brief.

5.1 Knots and Links

Here, we start with a basic overview of knot theory leading to the Jones polynomial.

- A knot is an embedding of a closed curve in $\mathbb{R}^3$; a circle is a trivial knot. A link is a finite family of disjoint union of knots. This implies that a knot is a link with a single closed curve. It is worth mentioning that every link is the closure of some braid.

- Knots are 3D objects but mostly studied through knot diagram obtained by projecting the 3D object onto a plane $\mathbb{R}^2$ just as depicted in figure 6. The non-trivial knots in figure 6 are examples showing that there is no unique way of embedding a closed curve in $\mathbb{R}^3$. Any closed curve can be knotted in different ways having different number of crossings. This leaves a question in knot theory: when are two knots, or perhaps two knot diagrams, equivalent?

Figure 6. The first two diagrams are non-trivial knots with 3 and 4 crossings respectively. The third diagram is the Hopf link.

- Existence of orientation preserving homeomorphism between two knot diagrams may provide equivalence of two knots. However, this is not always true and such homeomorphism is hard to find in practice.

- Two links are equivalent or isotopic if they differ by a finite sequence of local sequence of Reidemeister moves and an orientation- preserving homeomorphism of the plane. There are three types of Reidemeister moves as shown in figure 7. Any homeomorphism of the plane must preserve all crossing information.

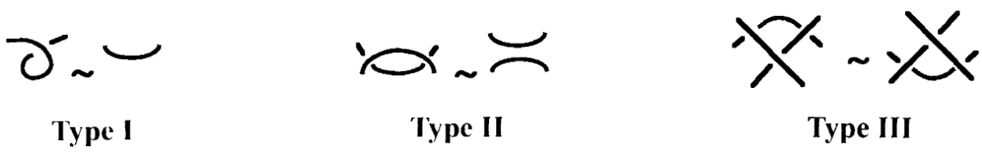

Figure 7. Three types of Reidemeister moves

- By orientation of a link, we means a choice of direction of trajectory around each component of the link. A cross change can be done in a diagram of an oriented link. Such crossing is a local modification of the type in figure 8. They are completely different from Reidemeister moves.

Figure 8. Two types of crossing change

- Any link can be unknotted by a finite number of cross changes. This cross changes allows for construction of link invariants. A link invariant assigns same object to links in the same isotopy class. An example of link invariant is the Jones polynomial.

Basically, the Jones polynomial is an invariant of link. Of all different model, Jones polynomial[16] is simpler to define in the light of Kauffman bracket polynomial. There are The Kauffman bracket is easier computed through skein relations.

Definition. *It is a function from unoriented link diagrams to Laurent polynomials in variable A using the following rules:*

1. $\langle \bigcirc \rangle = 1,$; $\bigcirc =$ *trivial link.*
2. $\langle L \cup \bigcirc \rangle = (-A^2 - A^{-2})\langle L \rangle$; $L =$ *link diagram.*
3. $\langle \times \rangle = A \langle \asymp \rangle + A^{-1} \langle)(\rangle .$

As an example, the computation of the bracket polynomial of Trefoil knot using skein relation is shown in figure 9. The bracket polynomial is invariant under type II and type III Reidemeister moves. Under type I move, the invariance is up to some pre-factors. This prefactor anomaly is as a result writhe.

Definition. *The writhe $w(L)$ of a diagram L of an oriented link is the sum of the signs of the crossing change defined according as the convention in figure 8.*

$w(L)$ is invariant under a type II or III Reidemeister move in L, but it does change by $+1$ or -1 under type I Reidemeister move in L. This is in fact the reason why bracket polynomial is not invariant due to some prefactors. It turns out that multiplication of bracket polynomial by $(-A)^{-3w(L)}$ cancels out all the prefactors left by Type I Reidemeister moves. This leads to the definition that the polynomial

$$X(L) = (-A)^{-3w(L)} \langle L \rangle, \tag{5.1}$$

is an invariant of the oriented link L. The Jones polynomials is a special case with variable $A = q^{-1/4}$.

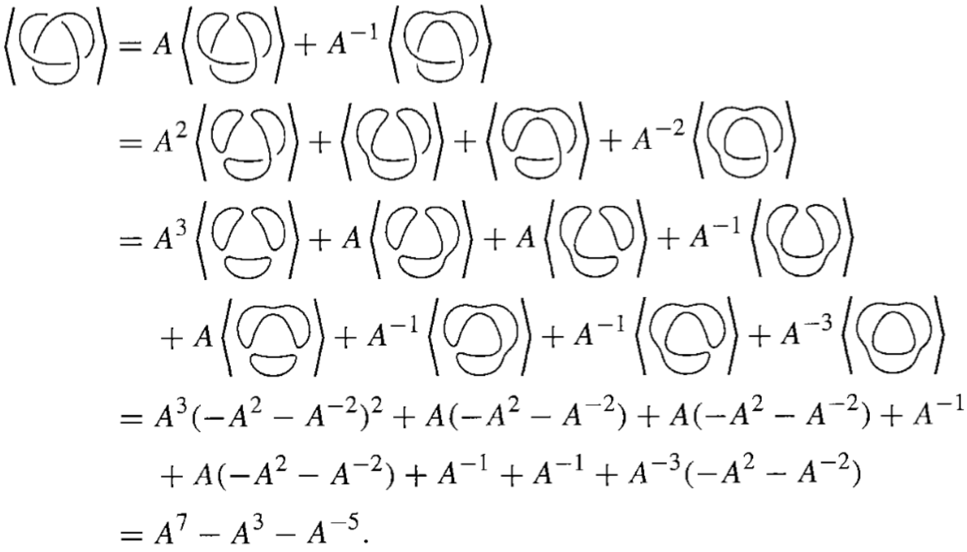

Figure 9. Bracket polynomial of Trefoil knot

Definition. *The Jones polynomial $V(L)$ of an oriented link L is a Laurent polynomial in variable $q^{1/2}$ assigned to an oriented link L defined by*

$$V(L) = (-A)^{-3w(L)} \langle L \rangle_{A^{-2}=q^{1/2}} \tag{5.2}$$

satisfying the following properties:

1. *V(trivial knot) =1*
2. *if L is isotopic to L', then $V(L) = V(L')$; the converse is not necessarily true.*
3. $q^{-1}V(L_+) - qV(L_-) + (q^{-1/2} - q^{1/2})V(L_0) = 0$, *where L_0, L_- and L_+ are Skein relation in Figure 10 showing identical oriented link diagrams but differ only at one crossing.*

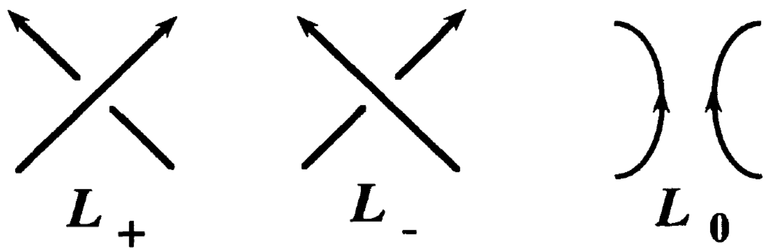

Figure 10. Skein relation

The Jones polynomial was discovered through a new representation of the Artin braid group. It has proved very useful in that, it has strong feature of distinguishing one link from another. This includes mirror images of link, which can be obtained for the Jones polynomial by replacing q with q^{-1}. Thus far, we have learn that any link can be unknotted by a finite number of crossing and that, the Jones polynomial allows us to evaluate the link invariant from the skein diagram.

6 Non-Abelian CS theory

Now, we shall return to Physics and embark on studying the Witten's approach to solving non-Abelian Chern-Simons theory. At the concluding part of section 4, we already highlighted the the basic procedure for this section. We shall now discuss them step-by-step. The first non-trivial step is the Quantization of CS theory on $\Sigma \times \mathbb{R}$. As we shall realize, the canonical approach for evaluating the Wilson loops will be based on constructing the Hilbert space on a Riemann surface Σ with punctures.

6.1 Quantization on $\Sigma \times \mathbb{R}$

We return back to the non-Abelian CS theory (3.8). In the euclidean theory, we can well write the Lagrangian density as

$$\mathcal{L} = \frac{k}{8\pi} \int_{\mathcal{M}} \epsilon^{ijk} \, \mathrm{Tr} \left(a_i \left(\partial_j a_k - \partial_k a_j \right) + \frac{2}{3} a_i \left[a_j, a_k \right] \right) \tag{6.1}$$

For now, we shall ignore the Wilson loop, and canonically quantize CS theory. Let us denotes the connection in the Σ-direction as $a_{i=1,2}$ and a_0 as the gauge connection in the $\mathbb{R}$-direction. Towards quantization, we recognize that there exist a natural gauge $a_0 = 0$ on $\Sigma \times \mathbb{R}$. With this gauge, the nonlinearity is tractable and the Lagrangian density reads

$$\mathcal{L} = \frac{k}{4\pi} \int_{\mathbb{R}} dt \int_{\Sigma} \epsilon^{ij} \, \mathrm{Tr} \left(a_i \frac{\partial}{\partial_t} a_j + a_0 f_{ij} \right). \tag{6.2}$$

In this way, a_i is a canonical conjugate of $\frac{k}{4\pi} \epsilon^{ij} a_j$. This implies that, in CS theory without the Wilson loops, components of gauge connection are canonical conjugate to one another. Furthermore, we recognize that a_0 is a Lagrangian multiplier that enforces a Gauss constraint $f_{ij} = 0$; as such, we are left with the quantization of contained system. Constrained systems can be quantized in two different but equivalent ways, namely,

- first canonically quantize and then enforces the Gauss constraint at the quantum level. In this way, we have the commutation relation as

$$[a_i^b(x), a_j^c(y)] = \frac{2\pi i}{k} \epsilon_{ij} \delta^{bc} \delta^2(x-y),$$

  subject to the Gauss constraint $\epsilon^{ij} f_{ij}^a = 0$ on the wave solution.

- Impose the Gauss constraint $\epsilon^{ij} f_{ij}^a = 0$ classically and then quantize the resulting degrees of freedom. So, we need first solve the constraint on Σ. This is equivalent to a problem of solving for the moduli space of flat connections on Σ for which curvature vanishes. A connection is expected to detail a way to do parallel transport in a principal bundle over Σ with structure group G_k. Therefore, for any closed loop in Σ, a connection determines a holonomy, i.e. the solutions of the constraint are parametrized by the holonomies of the gauge connection around a cycle in Σ. For a genus g surface Σ, and a gauge group G_k, the dimension of the moduli space of flat connection is $(2g-2)\, dim\, G_k$.

- We are particularly interested in $\Sigma = S^2$. However, there are no flat connection on S^2, thereby rendering quantization trivial with with just a unique state. As such, $dim\, \mathcal{H}_\Sigma = 1$.

- For a genus $g \geq 1$ surface Σ_g and gauge group $SU(2)_k$, the dimension of the Hilbert space is given by

$$dim\, \mathcal{H}_{\Sigma_g} = \left(\frac{k+2}{2}\right)^{g-1} \sum_{j=0}^{k} \left(\sin \frac{(j+1)\pi}{k+2}\right)^{2-2g} \tag{6.3}$$

This is the Verlinde formula[19] arising as a consequence of the fact that, the modular S-Matrix implementing modular transformation $\tau \to -\frac{1}{\tau}$ diagonalizes the fusion rule. A very important feature of this formula is that, it is an integer and a finite polynomial in k.

6.1.1 Quantization on Σ with punctures

So far, we have quantized the CS theory on Σ without the Wilson loop. On inclusion of the Wilson loop, the key insight is to recognise that the worldlines of particles will leave markpoints (punctures) of static charges at points ξ_i on the Riemann surface Σ as the loops pierce through it. Each point ξ_i corresponds to a representation R_i of the gauge group G_k. Quantization of CS theory in the presence of Wilson loop is therefore up to finding the Hilbert space $\mathcal{H}_{\Sigma,\xi_i,R_i}$ associated to the surface Σ with the choice of markpoint ξ_i corresponding to R_i. As the Wilson lines correspond to static non-abelian charges, the Gauss constraint is therefore modified as

$$\frac{k}{8\pi}\epsilon^{ij} f^c_{ij}(x) = \sum_{n=1}^{p} \delta^2\left(x - \xi_n\right) t^c_{(n)}, \tag{6.4}$$

where $t^c_{(r)}$, $c = 1, ..., dim\, G$ are the generator of the gauge group associated with static external charges placed at ξ_r. Following either of the earlier quantization approach result into rather quantization issues resolved as follows:

- the large k limit, i.e. $k \to \infty$ corresponds to the weak coupling limit of the CS theory, so that we can do calssical analysis.
- Dirac quantization criterion applies to each f^c_{ij}.
- for large k, Dirac quantisation cannot be applied alongside with non-trivial charge on the right hand side of the Gauss constraint.
- possible resolution is for all charges to sum up to zero. This is indeed the case as equal quantities of positive and negative charge is expected no matter how weak the coupling is. As such, the net charge is zero. In non-Abelian theory, this implies that all of the charges must be coupled with trivial representation of the gauge group.
- i.e. these are all trivial representation arising from the decomposition of $\bigotimes_{i=1}^{n} R_i$ in the large k limit and it is restricted to this.
- In the light of this resolution, the physical Hilbert space on $\Sigma = S^2$ with three marked points corresponding to representations R_i, R_j and $\bar{R}_k$ is spanned by the orthonormal states of all possible fusion channels $\mathcal{V}^k_{ij}$ whose dimension is $\sum_k \mathcal{N}^k_{ij}$.

In the spirit of quantization, there is a geometric construction of the Hilbert space according as the Borel-Weil-Bott theorem. They constructed the Hilbert space as the space of holomorphic sections of line bundle leading to a linear representation of G. In this formalism, all irreducible representation arise from such kind of construction, an understanding that become useful in quantization of Chern-Sirnons theory. The Hilbert space is given as

$$\mathcal{H}_{(\Sigma;\xi_1,R_1;\ldots;\xi_n,R_n)} = \bigotimes_{i=1}^{n} R_i^G = (R_1 \otimes \ldots \otimes R_n)^G. \tag{6.5}$$

- For $n = 0$, there is only the trivial representation on Σ and $dim\mathcal{H}_\Sigma = 1$.
- For $n = 1$, the Hilbert space is $\mathcal{H}_{\Sigma;\xi_1,R_1} = R_1^G$, so that $\mathcal{H}_{\Sigma;\xi_1,R_1} = 1$ if R is trivial and 0 otherwise.
- For $n = 2$, $dim\,\mathcal{H}_{\Sigma;\xi_1,R_1,\xi_2,R_2} = dim(R_1 \otimes R_2)^G = 1$ if R_2 is dual R_1 and 0 otherwise.
- For $n = 3$, the $dim\,\mathcal{H}_{\Sigma;\xi_i,R_i,\xi_j,R_j,\xi_k,R_k} = dim(R_i \otimes R_j \otimes R_k)^G = \mathcal{N}_{ij}^k$. The Hilbert space is spanned by the orthonormal states of all possible fusion channels.
- Let R be the representation of the gauge group $G = SU(N)$. With two Wilson loops, there are four marked points, i.e. $n = 4$, corresponding to representations R, R, $\bar{R}$ and $\bar{R}$. There are two distinct irreducible representations arising from decomposition $R \otimes R = E_1 \oplus E_2$. E_1 is the symmetric representation generated by $\frac{1}{2}(e_i \otimes e_j + e_j \otimes e_i)$, and E_2 is the antisymmetric representation generated by $\frac{1}{2}(e_i \otimes e_j - e_j \otimes e_i)$, so that we have

  $$R \otimes R \otimes \bar{R} \otimes \bar{R} = (E_1 \oplus E_2) \otimes (\bar{E}_1 \oplus \bar{E}_2).$$

  For SU(2), this implies that

  $$dim\,\mathcal{H}_{\Sigma;\xi_1,R_1,\xi_2,R_2,\xi_3,R_3,\xi_4,R_4} = dim\,(R \otimes R \otimes \bar{R} \otimes \bar{R})^G = 2, \tag{6.6}$$

  i.e. $SU(2)$ is generated by $E_1 \otimes \bar{E}_1$ and there are only two possible fusion outcomes. This is the central idea we shall use to explain skein relations in Jones polynomials.

6.2 Evaluation by TQFT

Before we embark on computation, we first establish some factorization properties which has potential of reducing evaluation of many loops to that of a single loop. This will follow from our earlier observation depicted in figure 5 and the TQFT axioms that follows it.

- **Factorization I:** Let $\mathcal{M}_1$ and $\mathcal{M}_2$ be two three manifolds and $\mathcal{M} = \mathcal{M}_1 \# \mathcal{M}_2$ be the connected sum (#) of two $\mathcal{M}_1$ and $\mathcal{M}_2$ along S^2. Following Atiyah's axiom, we deduce a factorization

  $$Z(M) \cdot Z(S^3) = Z(\mathcal{M}_1) \cdot Z(\mathcal{M}_1), \tag{6.7}$$

  where $Z(X)$ denotes the partition function over any three manifold X inside which knots may live. S^3 is the three sphere filling in S^2; it carries no knot and the partition function over it is $Z(S^3)$. (6.7) implies that

  $$\frac{Z(M)}{Z(S^3)} = \frac{Z(\mathcal{M}_1)}{Z(S^3)} \cdot \frac{Z(\mathcal{M}_1)}{Z(S^3)}. \tag{6.8}$$

- **Factorization II:** Given any link L, living in a three manifold $\mathcal{M}$, the expectation value of L is given as

  $$\langle L \rangle = \frac{Z(M, L)}{Z(M)}. \tag{6.9}$$

- **Factorization III:** Let $C_1, ..., C_n$ be n unlinked, unknotted loops living on the three sphere S^3. Following (6.8) and (6.9), then

$$\langle C_1, ..., C_n \rangle = \prod_{k=1}^{n} \langle C_k \rangle. \tag{6.10}$$

6.2.1 Unknotting Procedure

We shall now present SU(2) Chern-Simons theory and its connection with the Skein relation. We shall not repeat the entire story associating with figure 5 again, rather, we remind ourselves that our goal is to evaluate the partition function $Z(L)$ of a link L living on a three manifold $\mathcal{M}$, from which the expectation value $\langle W_L \rangle$ can be computed. Witten derived the rule for unknotting the link as follows:

- Embed a link L in a general three manifold $\mathcal{M}$ and let the link components (i.e. each Wilson loop) carry fundamental representation R of the $SU(2)$ gauge group. The link features a crossing configuration (see figure 11a). Imagine inserting a small sphere around the crossing configuration. The sphere would intersect the link at four marked points, so that it cuts $\mathcal{M}$ into two pieces corresponding to $\mathcal{M}_L$ and $\mathcal{M}_R$ in Figure 11b.

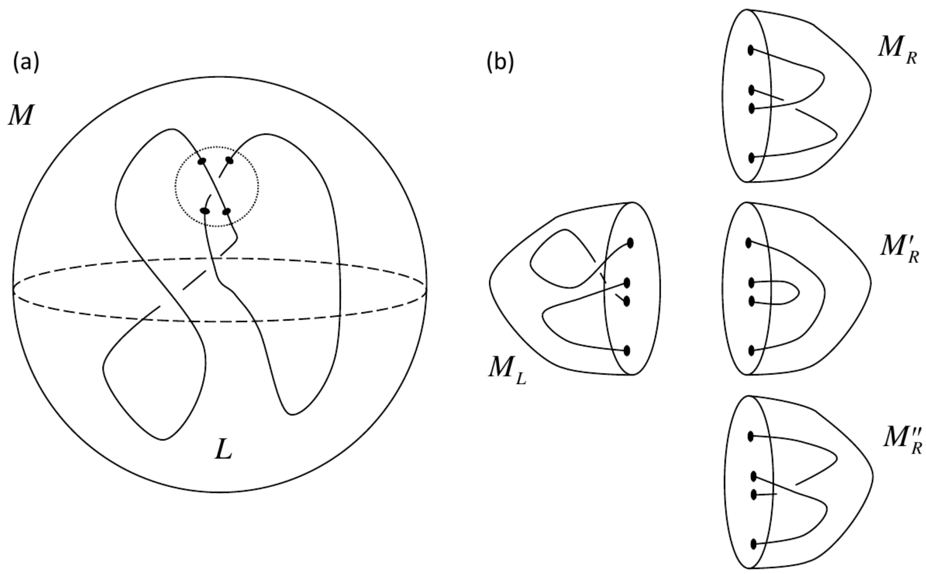

Figure 11. Cutting procedure of three manifold $\mathcal{M}$

- In this way, a part of the link L inside $\mathcal{M}_R$ has a single crossing and the other part is contained in $\mathcal{M}_L$. However, $\mathcal{M}_L$ and $\mathcal{M}_R$ shares same boundary $\Sigma = S^2$.
- We already quantized the theory on the boundary $\Sigma = S^2$. We therefore associate physical Hilbert spaces $\mathcal{H}_L$ and $\mathcal{H}_R$ with the boundaries of $\mathcal{M}_L$ and $\mathcal{M}_R$ respectively with $\mathcal{H}_L = \mathcal{H}_R^*$. We know the Hilbert spaces are two dimensional according as (6.6). Feynman path integral over $\mathcal{M}_L$ and $\mathcal{M}_R$ yields vectors $\langle \psi_{\mathcal{M}_L} | \in \mathcal{H}_L$ and $| \psi_{\mathcal{M}_R} \rangle \in \mathcal{H}_R$ respectively.
- The partition function then gives $Z(L) = \langle \psi_{\mathcal{M}_L} | U_f | \psi_{\mathcal{M}_R} \rangle$, where $U_f : \mathcal{H}_L \to \mathcal{H}_R$ is a unitary operator induced by orientation preserving homeomorphism f used in the gluing procedure.

- Again, $\mathcal{H}_L$ and $\mathcal{H}_R$ are two dimensional. Since a linear combination of three vectors in 2D vector space vanishes, then given $|\chi_{\mathcal{M}_R}\rangle, |\varphi_{\mathcal{M}_R}\rangle \in \mathcal{H}_R$, there is a linear relation

$$\alpha |\psi_{\mathcal{M}_R}\rangle + \beta |\chi_{\mathcal{M}_R}\rangle + \gamma |\varphi_{\mathcal{M}_R}\rangle = 0, \tag{6.11}$$

and as a consequence,

$$\begin{aligned} \alpha \langle\psi_{\mathcal{M}_L}| U_f |\psi_{\mathcal{M}_R}\rangle + \beta \langle\psi_{\mathcal{M}_L}| U_f |\chi_{\mathcal{M}_R}\rangle + \gamma \langle\psi_{\mathcal{M}_L}| U_f |\varphi_{\mathcal{M}_R}\rangle = 0, \\ \alpha Z(L) + \beta Z(L') + \gamma Z(L'') = 0, \end{aligned} \tag{6.12}$$

where L' and L'' are links associated with respective three manifolds $\mathcal{M}'$ and $\mathcal{M}''$. While we shall take $\mathcal{M}' = \mathcal{M} = \mathcal{M}''$, both L' and L'' carry worldline braiding different from that of L as a consequence of replacing three $\mathcal{M}_R$ with $\mathcal{M}'_R$ and $\mathcal{M}''_R$ respectively and application of braiding rule.

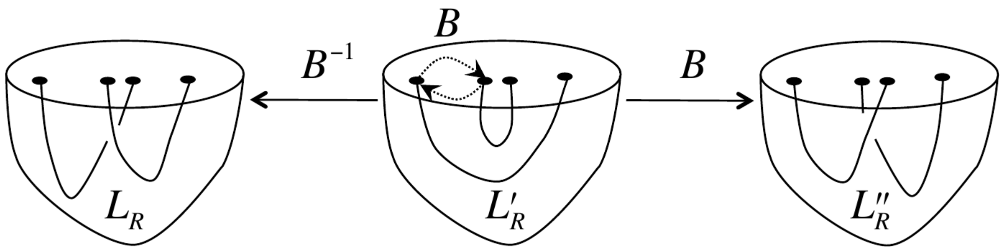

Figure 12. Braiding operation of a section of Wilson link living in $\mathcal{M}_R$

- As shown in Figure 12, clockwise exchange of the first two marked points on the boundary of $\mathcal{M}'_R$ yield the link configuration L_R, while the anticlockwise exchange yield the configuration L''_R. This clockwise and anticlockwise exchange are respectively implemented by braiding unitary operator $\mathcal{B}^{-1}$ and $\mathcal{B}$, so that state corresponding to L_R and L''_R are respectively given as

$$\begin{cases} |\psi_{\mathcal{M}_R}\rangle = \mathcal{B}^{-1} |\chi_{\mathcal{M}'_R}\rangle = \mathcal{B}^{-1} |\chi_{\mathcal{M}_R}\rangle \\ |\varphi_{\mathcal{M}_R}\rangle = \mathcal{B} |\chi_{\mathcal{M}'_R}\rangle = \mathcal{B} |\chi_{\mathcal{M}_R}\rangle . \end{cases} \tag{6.13}$$

- (6.12) is further represented in Figure 13. Indeed, this diagram represents an expression related to the Skein relation encountered in section 5 as a property of the Jones polynomial.

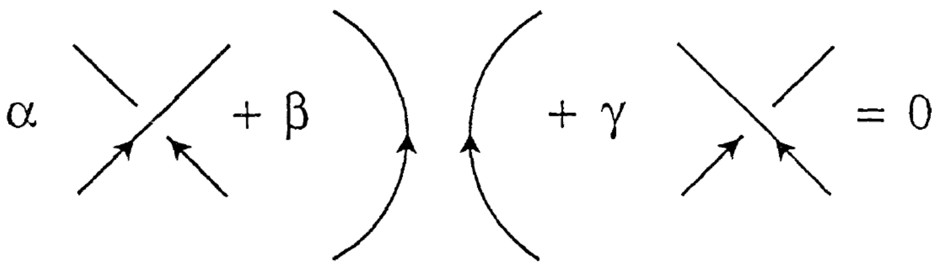

Figure 13. Skein-Recursion relation for links

- (6.12) is interpreted as follows. Considers three topologically equivalent closed contours corresponding to three links whose plane projections are identical outside a disc, and respectively have Skein relation inside the disc (see Figure 10 and 14). The expectation values of the three topologically equivalent closed contours weighted with α, β and γ is zero, i.e. $\alpha Z(L) + \beta Z(L') + \gamma Z(L'') = 0$ for three links L, L' and L''.

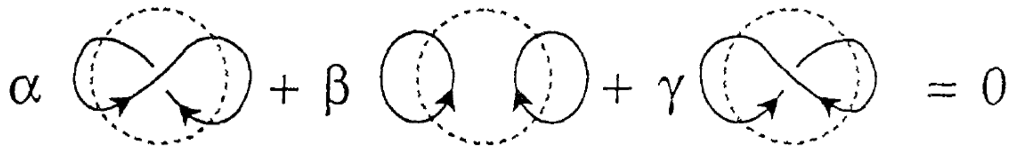

Figure 14. Skein relation for full links plane projection

- Now take $\mathcal{M} = S^3$ and denote the partition function of n unlinked, unknotted loops living on the three sphere S^3 as $Z(S^3; C_1, ..., C_n)$. Following Figure 14, then (6.12) sum up to

$$\alpha Z(S^3; C) + \beta Z(S^3; C^2) + \gamma Z(S^3; C) = 0.$$

Using the factorization properties earlier outlined, then

$$\alpha \frac{Z(S^3; C)}{Z(S^3; C)} + \beta \frac{Z(S^3; C^2)}{Z(S^3; C)} + \gamma \frac{Z(S^3; C)}{Z(S^3; C)} = 0$$

implies

$$\langle C \rangle = -\frac{\alpha + \gamma}{\beta}. \tag{6.14}$$

- For $SU(N)_k$, a braiding operator B whose eigenvalues are

$$\lambda_1 = \exp\left(\frac{i\pi(-N+1)}{N(N+k)}\right), \quad \lambda_2 = -\exp\left(\frac{i\pi(N+1)}{N(N+k)}\right)$$

can be used to evaluate expectation value $\langle C \rangle$ with

$$\begin{aligned} \alpha &= -\exp\left(\frac{2\pi i}{N(N+k)}\right), \\ \beta &= -\exp\left(\frac{i\pi(2-N-N^2)}{N(N+k)}\right) + \exp\left(\frac{i\pi(2+N-N^2)}{N(N+k)}\right), \\ \gamma &= \exp\left(\frac{2\pi i(1-N^2)}{N(N+k)}\right). \end{aligned}$$

- In fact, by introducing variable $q = \exp\left(\frac{2\pi i}{N+k}\right)$, the very property of Jones polynomial

$$-q^{N/2} L + \left(q^{1/2} - q^{-1/2}\right) L' + q^{-N/2} L'' = 0, \tag{6.15}$$

realizes the expectation value for the single loop as

$$\langle C \rangle = -\frac{\alpha + \gamma}{\beta} = \frac{q^{N/2} - q^{-N/2}}{q^{1/2} - q^{-1/2}}, \tag{6.16}$$

- While this computation is for a single Wilson loop, the same procedure generalizes to more many complicated loops. This immediately suggest the interpretation of the partition function $Z(M; L, R)$ of the $SU(N)_k$ Chern-Simons theory as nothing but the Jones polynomial, evaluated at $q = \exp\left(\frac{2\pi i}{N+k}\right)$, and the problem is solved.

Conclusion

Our study cut across many interesting physical ideas, each of which are very enlightening and may worth studying independently. These include some progresses in planar condensed matter physics leading to non-Abelian anyons, and then to finding a good topological quantum field theory (TQFT) to describe those physics. While the studies in relation to understanding these rich concepts is inexhaustive, we have been able to, at least, achieve a primary aim of recognizing the $2+1$-dimensional Chern-Simons gauge theory as a good example TQFT exhibiting most important features of the underlying physics of the quantum hall system and the non-Abelian anyons.

The study also exposed us to the realization that, there is really lots of stories and ideas about quantum Chern-Simons invariants in topology, and physics. Indeed, we have learnt a very important lesson, in that, given a link $L = \cup_{C_i} \subset S^3$ in the representation R_i of the $SU(N)$ Chern-Simons level k theory, the partition function,

$$Z(S^3; L, R_i) = \int \mathcal{D}a \exp\left(\frac{ik}{4\pi}\int_{S^3} \mathrm{Tr}(a \times da + \frac{2}{3} a \times a \times a)\right) \prod_i \mathrm{Tr}_{R_i} \mathcal{P} \exp \int_{C_i} a,$$

is the Jones polynomial evaluated at variable $q = \exp\left(\frac{2\pi i}{N+k}\right)$. This is Witten's novel achievement. As such, it has provided another exchange between mathematics and physical studies.

Suffices to mention here that the Chern-Simons terms have been accommodated into some other theories. Due to the gauge principle it exhibits, CS effective modification connects interesting areas of physics studies such as particle physics, String Theory Loop Quantum Gravity and Cosmology. There is so much more to groak.

Acknowledgments

I thank Richard Eager for making this topic available for my study and seminar work. My knowledge of the topic improves greatly through some useful discussion with Ingmar Saberi.

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