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THE THEORY OF THE COLLATZ PROCESS AND THE METHOD OF DYNAMICAL BALLS

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ABSTRACT. In this paper we introduce and develop the theory of the Collatz process and the method of dynamical balls. We leverage this theory to study the Collatz conjecture. This theory also has a subtle connection with the infamous problem of the distribution of Sophie germain primes. We also provide several formulation of the Collatz conjecture in this language. Furthermore, we introduce and develop the notion of dynamical systems induced by a fixed $a \in \mathbb{N}$ and their associated induced dynamical balls. We develop tools to study problems requiring to determine the convergence of certain sequences generated by iterating on a fixed integer.

1. Introduction and motivation

Recall the Collatz function, the arithmetic function of the form

Definition 1.1. Let $a \in \mathbb{N}$, then the Collatz function is the piece-wise function

$$\mathcal{C}(a) = \begin{cases} \frac{a}{2} & \text{if} \quad a \equiv 0 \pmod 2 \\ 3a+1 & \text{if} \quad a \equiv 1 \pmod 2. \end{cases}$$

Then Collatz conjecture, which is one of the acclaimed hardest but easy to state problems is the assertion that

Conjecture 1.1. Let $\mathcal{C}$ be the Collatz function, then $\min\{\mathcal{C}^s(b)\}_{s=0}^{\infty} = 1$ for any $b \in \mathbb{N}$.

The conjecture has long been studied and hence the vast literature and surveys concerning the study. For instance the problem has been given a fair treatment in the following surveys [4], [2], [3]. Motivated by this problem we introduce the subject of the Collatz process. We develop this theory and it turns out incidentally that it is connected to other open problems such as the problem concerning the distribution of the Sophie germain primes.
It needs to be said that the classical problem of deciding on the convergence of a given sequence is generally in principle not a hard problem. However, the difficulty may arise from the how the terms in the sequence are generated. A typical example of a sequence whose convergence may be difficult to determine is the Collatz sequence

$$f(n), f^2(n), f^3(n), \ldots, f^k(n), \ldots$$

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where $f^s = f^{s-1} \circ f = f \circ f^{s-1}$ and

$$f(n) := \begin{cases} \frac{n}{2} & \textbf{if} \quad n \equiv 0 \pmod 2 \\ 3n+1 & \textbf{if} \quad n \equiv 1 \pmod 2 \end{cases}.$$

Collatz conjecture [5] is the problem requiring to decide on the convergence of the system for all $n \in \mathbb{N}$. Another problem of possibly similar difficulty is the problem to determine the convergence of the juggler sequence introduced by Pickover [6]

$$g(n), g^2(n), g^3(n), \ldots, g^k(n), \ldots$$

with $g^s = g \circ g^{s-1} = g^{s-1} \circ g = g \circ g \cdots \circ g$ (s rimes) for all $n \in \mathbb{N}$, where

$$g(n) := \begin{cases} \lfloor n^{\frac{1}{2}} \rfloor & \textbf{if} \quad n \equiv 0 \pmod 2 \\ \lfloor n^{\frac{3}{2}} \rfloor & \textbf{if} \quad n \equiv 1 \pmod 2 \end{cases}.$$

These problems are widely believed to be difficult and forbidden, given that there are currently no viable tool for making ample progress [5], [4].
In this paper, we generalize these problems by introducing the notion of dynamical systems and their corresponding dynamical balls. We develop some tools to study problems of the form above.

2. **Modified Collatz function and the Collatz process**

In this section we introduce s slight variant of the Collatz function and introduce the notion of the Collatz process. We introduce the notion of the backward Collatz process and the generator of the Collatz process.

Definition 2.1. Let $a \geq 1$, then the Collatz function is the piece-wise function

$$f(a) = \begin{cases} \frac{a}{2} & \text{if} \quad a \equiv 0 \pmod 2,\ a > 1 \\ 3a+1 & \text{if} \quad a \equiv 1 \pmod 2,\ a > 1 \\ 1 & \text{If} \quad a = 1. \end{cases}$$

Definition 2.2. Let f be the Collatz function and let $a \geq 1$. Then by the Collatz process on a, we mean the sequence $\{f^s(a)\}_{s=1}^{\infty}$ for $s \in \mathbb{N}$. The sequence $\{\mathrm{Inf}\{f^{-s}(a)\}\}_{s=1}^{\infty}$ is the backward Collatz process and a is said to be the generator of the Collatz process if each $a_n \in \{\mathrm{Inf}\{f^{-s}(a)\}\}_{s=1}^{\infty}$ is of the same parity.

Proposition 2.1. *Let f be the Collatz function with the corresponding Collatz process $\{f^s(b)\}_{s=1}^{\infty}$. If b is the generator, then each $a_n \in \{\mathrm{Inf}\{f^{-s}(b)\}\}_{s=1}^{\infty}$ must satisfy*

$$a_n \equiv 0 \pmod 2.$$

Proof. Let f be the Collatz function with generator $b \in \mathbb{N}$. First we see that $f^{-1}(b) \not\equiv 1 \pmod 2$; Otherwise it would mean $f^{-2}(b) \equiv 0 \pmod 2$ contradicting the assumption that b is the generator of the process. It suffices to consider the case $m \geq 2$. Suppose for the sake of contradiction that there exist some $a_m \in \{\mathrm{Inf}\{f^{-s}(b)\}\}_{s=1}^{\infty}$; in particular, let $a_m = f^{-m}(b)$ such that $a_m \equiv 1 \pmod 2$ for $m \geq 2$. Then under the Collatz process, we must have

$$3f^{-m}(b) = f^{-(m-1)}(b) - 1.$$

It follows that $f^{-(m-1)}(b) \equiv 0 \pmod 2$, thereby contradicting the underlying assumption that b is the generator of the process. □

Proposition 2.2. *Let f be the Collatz function with the corresponding Collatz process $\{f^s(b)\}_{s=1}^{\infty}$. If $b \in \mathbb{N} \setminus \{1\}$ is the generator of the process, then $b \notin \{f^s(b)\}_{s=1}^{\infty}$.*

Proof. Let f be the Collatz function with the corresponding process $\{f^s(b)\}_{s=1}^{\infty}$ with generator $b \in \mathbb{N} \setminus \{1\}$. Suppose on the contrary that $b \in \{f^s(b)\}_{s=1}^{\infty}$. Then it follows that there exist some $s \geq 1$ such that $f^s(b) = b$. Thus we obtain the following chains of equality

$$b = f^s(b) = f^{2s}(b) = f^{3s}(b) = \ldots$$

for some $s \geq 1$. It follows from Proposition 2.1 that each $a_n \in \{f^s(b)\}_{s=m;m\geq 1}^{\infty}$ must satisfy the parity condition $a_n \equiv 0 \pmod 2$, since b is the generator of the process. This is not true since f is the Collatz function. □

Remark 2.3. Next we prove the unicity of generators of the Collatz process.

Proposition 2.3. *The generator of any Collatz process is unique.*

Proof. Let f be the Collatz function with the corresponding processes $\{f^s(a)\}_{s=1}^{\infty} = \{f^s(b)\}_{s=1}^{\infty}$, where $a, b \in \mathbb{N}$ are the two generators such that $a \neq b$ with $a, b > 1$. Then it follows that for any $m \geq 1$, there exist some $r > 1$ such that we have $f^m(b) = f^r(a)$. Without loss of generality, we let $m > r$ so that we have $f^{m-r}(b) = a$. It follows that $a \in \{f^s(b)\}_{s=1}^{\infty}$. Appealing to Proposition 2.2, then $a \notin \{f^s(a)\}_{s=1}^{\infty}$ and it follows that $\{f^s(a)\}_{s=1}^{\infty} \neq \{f^s(b)\}_{s=1}^{\infty}$, thereby contradicting the assumption that a, b are two distinct generators of the process. □

It is very important to point out that a Collatz process may or may not have a generator. If a Collatz process has a generator, then we say the generator is finite; On the other hand, If it has no generator then we say the generator is infinite. Next we expose the parity of a generator of a Collatz process.

Proposition 2.4. *Let f be the Collatz function with the corresponding process $\{f^s(b)\}_{s=1}^{\infty}$ for $b \in \mathbb{N}$. If $b \neq 1$ is the generator of the process, then $b \equiv 1 \pmod 2$.*

Proof. Let f be the Collatz function with the corresponding process $\{f^s(b)\}_{s=1}^{\infty}$ and suppose $b \in \mathbb{N}$ is the generator of the process. Then by definition each $a_n \in \{\mathrm{Inf}\{f^{-s}(b)\}\}_{s=1}^{\infty}$ has the same parity and must satisfy $a_n \equiv 0 \pmod 2$. Suppose on the contrary that $b \equiv 0 \pmod 2$, then we choose $m \geq 1$ for $m = \mathrm{Inf}(s)_{s=1}^{\infty}$ such that $f^m(b) \equiv 1 \pmod 2$. Then it follows that each

$$a_n \in \{\mathrm{Inf}\{f^{-s}(b)\}\}_{s=1}^{\infty} \cup \{b, f(b), f^2(b), \ldots, f^{m-1}(b)\}$$

has the same parity. It follows that $f^m(b)$ is the generator of the process $\{f^s(b)\}_{s=m+1;m\geq 1}^{\infty}$. Since $\{f^s(b)\}_{s=m+1;m\geq 1}^{\infty} \subset \{f^s(b)\}_{s=1}^{\infty}$, It follows that for some $f^r(b) \in \{f^s(b)\}_{s=m+1;m\geq 1}^{\infty}$, there exist some $f^t(b) \in \{f^s(b)\}_{s=1}^{\infty}$ such that $f^r(b) = f^t(b)$. It then follows that $f^k(b) = b$ for $k \geq 1$. This relation is absurd under the Collatz function. □

2.1. **The order and index under the Collatz process.** In this section we introduce the notion of the order and the index of positive integers under the Collatz process. We study the convergence and the divergence of the Collatz process. We launch the following terminology to aid our inquiry.

Definition 2.4. Let f be the Collatz function and $a > 1$, then the order of a under the Collatz process is the least value of m such that $f^m(a) = 2^k$. The value of k is the index of a under the Collatz process. The number a is said to have finite order and a finite index if and only if it converges under the Collatz process. Otherwise we say it diverges under the Collatz process. We denote by $\tau_f(a)$ and $\mathrm{Ind}_f(a)$ the period and the index of a under the Collatz process. In the case a diverges under the process, then $\tau_f(a) = \infty$ and $\mathrm{Ind}_f(a) = \infty$.

In light of definition 2.4, the Collatz conjecture can be restated in the following manner:

Conjecture 2.1 (Collatz)**.** Let f be the Collatz function and $\{f^s(a)\}_{s=1}^{\infty}$ for $a \in \mathbb{N}$ be a Collatz process, then $\tau_f(a) < \infty$.

The above conjecture can also be expressed in a more quantitative form. In other words, It suffice to resolve the Collatz conjecture by showing that

Conjecture 2.2 (Collatz)**.** Let f be the Collatz function and $\{f^s(b)\}_{s=1}^{\infty}$ an arbitrary Collatz process. Then

$$\sum_{s=1}^{\infty} \log(f^s(b)) < \infty.$$

Proposition 2.5. *Let f be the Collatz function and let $a > 1$ with $\Omega(a) = 2$ such that $a \equiv 0 \pmod 2$, then*

$$f(r) - 1 = 3f(a)$$

where $a = 2r$ with $r \equiv 1 \pmod 2$.

Proof. Since a is even, it follows from definition 2.1 that the right hand side must be $3r$. Under the condition that $\Omega(a) = 2$ with $r \equiv 1 \pmod 2$, the result follows by definition 2.1. □

Remark 2.5. Next we show that primes in a certain congruence class should, by necessity, have large order in as much as their index under the Collatz process is large.

Theorem 2.6. *Let f be the Collatz function and $p > 3$ be a prime such that $p \equiv 3 \pmod 4$. If $\mathrm{Ind}_f(p) > 1$, then $\tau_f(p) > 1$.*

Proof. Let $p > 3$ be a prime, then under the Collatz process 2.4, it follows that $f^{\tau_f(p)}(p) = 2^{\mathrm{Ind}_f(p)}$. Suppose on the contrary $\tau_f(p) = 1$, then under the assumption $\mathrm{Ind}_f(p) > 1$ it follows that $2^{\mathrm{Ind}_f(p)} + 1 \equiv 1 \pmod 4$. Then it must certainly be that $2^{\mathrm{Ind}_f(p)} - 1 \equiv 3 \pmod 4$, so that under the Collatz process we have

$$3p \equiv 3 \pmod 4$$
$$\iff p \equiv 1 \pmod 4$$

thereby contradicting the residue class of the prime $p > 3$. □

Remark 2.7. Next we establish a converse of Theorem 2.6 in the following proposition.

Proposition 2.6. *Let f be the Collatz function with the corresponding convergent Collatz process $\{f^s(b)\}_{s=1}^{\infty}$ for $b \in \mathbb{N}$. If $\tau_f(b) \geq 2$, then $\mathrm{Ind}_f(b) > 1$.*

Proof. Let f be the Collatz function with a corresponding convergent Collatz process $\{f^s(b)\}_{s=1}^{\infty}$. Let $\tau_f(b) \geq 2$ and suppose on the contrary that $\mathrm{Ind}_f(b) = 1$, then we can write $f^{\tau_f(b)}(b) = 2$. Since $\tau_f(b) \geq 2$, we can write $f^{\tau_f(b)-1}(b) = f^{-1}(2) = 4$. This contradicts the minimality of $\tau_f(b)$, since $f^{\tau_f(b)-1}(b) \in \{f^s(b)\}_{s=1}^{\infty}$. □

2.2. **Relative speed of the Collatz process.** In this section we introduce the notion of the relative speed of a Collatz process.

Definition 2.8. Let f be the Collatz function with the corresponding Collatz process $\{f^s(a)\}_{s=1}^{\infty}$. Then by the speed of the j th Collatz process relative to the k th Collatz process, we mean the expression

$$\nu(f^j(a), f^k(a)) = \frac{|f^k(a) - f^j(a)|}{|k-j|}.$$

The Collatz conjecture can also be framed in the language of the relative speed of the Collatz process as

Conjecture 2.3. Let f be the Collatz function with the corresponding Collatz process $\{f^s(b)\}_{s=1}^{\infty}$ for $b \in \mathbb{N}$, then there exist some $1 \leq j < k$ such that $\nu(f^j(b), f^k(b)) = 2^r$ for some $r \in \mathbb{N}$.

It follows from definition 2.8 that the relative speed of the Collatz process must satisfy the inequality

$$|f^k(b) - f^j(b)| = |k-j|\nu(f^j(b), f^k(b)) \leq f^j(b) + f^k(b).$$

Thus the Collatz conjecture is equivalent to establishing the inequality

Conjecture 2.4. Let f be the Collatz function with the corresponding process $\{f^s(b)\}_{s=1}^{\infty}$, then there exist some $k \geq 1$ such that the inequality is valid

$$2^r \leq \nu(f^{k+1}(b), f^k(b)) \leq 2^m$$

for some $m, r \in \mathbb{N}$.

2.3. **The sub-Collatz process.** In this section we introduce the notion of the sub-Collatz process. We establish a relationship between the order and the index of a number under the assumption that the Collatz process converges. We launch the following language.

Definition 2.9. Let f be the Collatz function. Then the Collatz process $\{f^t(a)\}_{t=1}^{\infty}$ is said to be a sub-Collatz process of the Collatz process $\{f^s(b)\}_{s=1}^{\infty}$ for $s, t \in \mathbb{N}$ if $\{f^t(a)\}_{t=1}^{\infty} \subseteq \{f^s(b)\}_{s=1}^{\infty}$. It is said to be proper if $\{f^t(a)\}_{t=1}^{\infty} \subset \{f^s(b)\}_{s=1}^{\infty}$. The Collatz process $\{f^s(b)\}_{s=1}^{\infty}$ is said to be full if b is the generator of the process.

Remark 2.10. Next we state a result that indicates that Collatz processes are indistinguishable once they overlap.

Proposition 2.7. *Let* $\{f^s(a)\}_{s=1}^{\infty}$ *and* $\{f^s(b)\}_{s=1}^{\infty}$ *be full Collatz processes. If* $\{f^s(a)\}_{s=1}^{\infty} \cap \{f^s(b)\}_{s=1}^{\infty} \neq \emptyset$*, then* $\{f^s(a)\}_{s=1}^{\infty} = \{f^s(b)\}_{s=1}^{\infty}$ *and* $a = b$.

Proof. Let $\{f^s(a)\}_{s=1}^{\infty}$ and $\{f^s(b)\}_{s=1}^{\infty}$ be full Collatz processes and suppose $\{f^s(a)\}_{s=1}^{\infty} \cap \{f^s(b)\}_{s=1}^{\infty} \neq \emptyset$. It follows that there exist some $t, m \geq 1$ such that $f^t(a) = f^m(b)$. Thus we obtain the following chains of equality

$$f^{(t+1)}(a) = f^{(m+1)}(b), \quad f^{(t+2)}(a) = f^{(m+2)}(b), \ldots f^{t+j}(a) = f^{(m+j)}(b)$$

for all $t \geq 1$. It follows that $\{f^s(a)\}_{s=t}^{\infty} = \{f^s(b)\}_{s=m}^{\infty}$. It follows that $\{f^s(a)\}_{s=1}^{\infty} \subseteq \{f^s(b)\}_{s=1}^{\infty}$ and $\{f^s(b)\}_{s=1}^{\infty} \subseteq \{f^s(a)\}_{s=1}^{\infty}$. This implies that $\{f^s(a)\}_{s=1}^{\infty} = \{f^s(b)\}_{s=1}^{\infty}$. Since the processes $\{f^s(a)\}_{s=1}^{\infty}$ and $\{f^s(b)\}_{s=1}^{\infty}$ are full, It follows by definition 2.9 that a and b are two generators of the process and by appealing to Proposition 2.3, It follows that $a = b$. □

Proposition 2.8. *Let* f *be the Collatz function with the corresponding full process* $\{f^s(b)\}_{s=1}^{\infty}$*. If the generator is trivial, then each* $a_n \in \{\mathrm{Inf}\{f^{-s}(b)\} - 1\}$ *must be of the form* $c_n = 2^n - 1$.

Proof. Let f be the Collatz function and suppose $\{f^s(b)\}_{s=1}^{\infty}$ is a full process. Then it follows that b is the generator of the process and $\{\mathrm{Inf}\{f^s(b)\}\}_{s=1}^{\infty}$ is the backward Collatz process, so that for each $a_n \in \{\mathrm{Inf}\{f^s(b)\}\}_{s=1}^{\infty}$ satisfies the parity condition $a_n \equiv 0 \pmod 2$. Since $\frac{f^{m+1}(b)}{2} = f^{-m}(b)$ for $m \geq 1$, It follows that $a_n = 2^{n-1}f^{-1}(b)$. Since the process is trivial, It follows that $b = 1$ and $f^{-1}(1) = 2$ and the result follows immediately. □

There does appear an important relation between the index and the order of primes generators. In light of this we state the following conjecture

Conjecture 2.5. Let f be the Collatz function and $\{f^s(b)\}_{s=1}^{\infty}$ be a full Collatz process. If $\{f^s(b)\}_{s=1}^{\infty}$ converges, then b is prime if and only if

$$\mathrm{Ind}_f(b) = \tau_f(b) + 1.$$

2.4. **Distribution of the Collatz process.** In this section we study the local and the global distribution of any Collatz process. We focus our study to the existence of primes in any Collatz process under a given generator.

Proposition 2.9. *Let* f *be the Collatz function with the corresponding full process* $\{f^s(b)\}_{s=1}^{\infty}$ *for* $b > 1$*. If* $f^2(b)$ *is prime, then* $2f^2(b) - 1$ *cannot be prime.*

Proof. Let f be the Collatz function with the corresponding full Collatz process $\{f^s(b)\}_{s=1}^{\infty}$. Then b is the generator and it follows from Proposition 2.4 that $b \equiv 1 \pmod 2$, so that $f(b) \equiv 0 \pmod 2$. Then it is certainly the case that $f^2(b) = \frac{f(b)}{2} = \frac{3b+1}{2}$. The claim follows from this relation. □

Remark 2.11. Next we expose the Theory to the infamous problem concerning the distribution of the Sophie germain primes. It reduces the problem entirely to knowing the existence and the distribution of consecutive primes in the unit left translates of the backward Collatz process.

Theorem 2.12. *Let* f *be the Collatz function with the corresponding Collatz process* $\{f^s(b)\}_{s=1}^{\infty}$*. Let* $\{f^s(b)\}_{s=1}^{\infty}$ *be a full process and* $f^{-k}(b), f^{-(k+1)}(b) \in \{\mathrm{Inf}\{f^{-s}(b)\}_{s=1}^{\infty}$*, the backward Collatz process. If* $f^{-k}(b) - 1, f^{-(k+1)}(b) - 1$ *are both prime, then*

$f^{-k}(b)-1$ must be a Sophie germain prime. Moreover there are infinitely many Sophie germain primes if there are infinitely many consecutive primes in $\{\mathrm{Inf}\{f^{-s}(b)\}-1\}_{s=1}^{\infty}$.

Proof. Let f be the Collatz function with the corresponding Collatz process $\{f^s(b)\}_{s=1}^{\infty}$. Since the process $\{f^s(b)\}_{s=1}^{\infty}$ is full, It follows that b must be the generator of the process. It follows from Proposition 2.1 each $f^{-m}(b) \in \{\mathrm{Inf}\{f^{-s}(b)\}_{s=1}^{\infty}$ must satisfy the parity condition $f^{-m}(b) \equiv 0 \pmod 2$. Then under the Collatz process, we obtain the following increasing sequence

$$\ldots > f^{-(m+2)}(b) > f^{-(m+1)}(b) > f^{-m}(b) > \ldots > f^{-1}(b)$$

with each term satisfying the equality $f^{-(j+1)}(b) = 2f^{-j}(b) \Longleftrightarrow f^{-(j+1)}(b) - 1 = 2(f^{-j}(b) - 1) + 1$. Under the assumption that $f^{-k}(b) - 1, f^{-(k+1)}(b) - 1$ are both prime, then the result follows immediately. If there are infinitely many consecutive primes $f^{-k}(b) - 1, f^{-(k+1)}(b) - 1 \in \{\mathrm{Inf}\{f^{-s}(b)\} - 1\}_{s=1}^{\infty}$, then it will follow that there are infinitely many Sophie germain primes. □

Theorem 2.12 relates the Collatz process to the problem of the distribution of Sophie germain primes. Indeed it sets out the idea that studying the problem on the set of integers is in some way silly. Rather it will much more technique convenient to study the unit left translate of the sequence arising from the backward Collatz process.

Remark 2.13. It turns out that certain Collatz process which when undergo a unit left translate has most of its elements being prime. This notion is exemplified in the following result.

Theorem 2.14. *Let f be a Collatz function, with the corresponding Collatz process $\{f^s(b)\}_{s=1}^{\infty}$. If the process is full, then $\mathcal{M} = \{\mathrm{Inf}\{f^{-s}(b)\}-1\}_{s=1}^{\infty}$ contains a prime.*

Proof. Let f be the Collatz function with the corresponding full process $\{f^{-s}(b)\}_{s=1}^{\infty}$. Then it follows that b is the generator of the process. It follows that the sequence $\{\mathrm{Inf}\{f^{-s}(b)\}_{s=1}^{\infty}$ is the backward Collatz process. By Proposition 2.4, each $a_n \in \{\mathrm{Inf}\{f^{-s}(b)\}_{s=1}^{\infty}$ must satisfy the parity condition $a_n \equiv 0 \pmod 2$ and additionally that

$$a_n = 2^{n-1}f^{-1}(b) - 1$$

for $n \geq 1$ with $f^{-1}(b) \in \{\mathrm{Inf}\{f^{-s}\}_{s=1}^{\infty}$. It follows that a_n must be prime for some $n \geq 1$. □

Conjecture 2.6. Let f be a Collatz function, with the corresponding Collatz process $\{f^s(b)\}_{s=1}^{\infty}$. If the process is full, then $\mathcal{M} = \{\mathrm{Inf}\{f^{-s}(b)\} - 1\}_{s=1}^{\infty}$ contains infinitely consecutive primes and

$$\lim_{s \longrightarrow \infty} \frac{\#(\mathcal{M} \cap \rho)}{\#\mathcal{M}} = 1$$

where ρ is the set of all primes.

Conjecture 2.6 is equivalent to the problem of deciding on the distribution of the Sophie germain prime, which is still an unsolved problem in the subject and we do not pursue in this paper. The Collatz process with a given generator exhibits other stunning and subtle properties in terms of the primality of the terms. In light of this we make the following conjectures

Conjecture 2.7. Let $p>2$ be prime with the corresponding Collatz process such that $p\equiv 3 \pmod 4$, then there exist $n\in \{f^s(p)\}_{s=1}^{\infty}$ such that $\mu(n)\neq 0$.

Conjecture 2.8. If $\{f^s(b)\}_{s=1}^{\infty}$ is a full Collatz process, then there exist an odd prime $p\in \{f^s(b)\}_{s=1}^{\infty}$.

Conjecture 2.9. Let f be the Collatz function with the corresponding Collatz process $\{f^s(a)\}_{s=1}^{\infty}$. If the process $\{f^s(a)\}_{s=1}^{\infty}$ is full, then $\Omega(a)\leq 2$.

3. Dynamical systems induced by sequences

In this section, we introduce and develop the notion of dynamical systems induced by a fixed $a\in \mathbb{N}$ and their associated induced dynamical balls. We develop tools to study problems requiring to determine the convergence of certain sequences generated by iterating on a fixed integer.

Definition 3.1. Let $f:\mathbb{N}\longrightarrow \mathbb{N}$. Then by the first k **dynamical** system induced by f on $a\in \mathbb{N}$, we mean the sequences generated by the system of iterations

$$f(a),f^2(a),f^3(a),\ldots,f^k(a)$$

where $f^s(a)=f\circ f^{s-1}(a)$ with $f^0(a)=a$ and equipped with the self generative energy

$$\mathcal{E}_a(f,f^2,\ldots,f^k):=\prod_{i=1}^{k}f^i(a)$$

with corresponding sequence of balls

$$\mathcal{B}_{f(a)}(a),\mathcal{B}_{f^2(a)}(a),\ldots,\mathcal{B}_{f^k(a)}(a)$$

where $f^s(a)$ is the radius of the s^{th} ball in the sequence and a the center of each ball. We call each ball in the sequence generated in this manner a **dynamical** ball. In other words, we say f induces a k dynamical system on $a\in\mathbb{N}$. We call $f^s(a)$ for $1\leq s\leq k$ the s^{th} dynamical system and $\mathcal{B}_{f^s(a)}(a)$ the s^{th} dynamical ball. We say the s^{th} dynamical system has an upward **measure** relative to the $(s-1)^{th}$ dynamical system if $f^s(a)>f^{s-1}(a)$. On the other hand, we say it has a downward **measure** relative to the $(s-1)^{th}$ dynamical system if $f^{s-1}(a)>f^s(a)$. Similarly, we say the s^{th} dynamical ball $\mathcal{B}_{f^s(a)}(a)$ is **inflated** relative to the ball $\mathcal{B}_{f^{s-1}(a)}(a)$ if $f^s(a)>f^{s-1}(a)$. We say it is **deflated** relative to the dynamical ball $\mathcal{B}_{f^{s-1}(a)}(a)$ if $f^{s-1}(a)>f^s(a)$. In the situation where $f^s(a)=f^{s-1}(a)$ then we say the s^{th} dynamical system is **stable** relative to the $(s-1)^{th}$ dynamical system, and the s^{th} dynamical ball $\mathcal{B}_{f^s(a)}(a)$ is **stable** relative to the $(s-1)^{th}$ dynamical ball $\mathcal{B}_{f^{s-1}(a)}(a)$.

Proposition 3.1. *Let f and g induce k dynamical system on $a\in\mathbb{N}$. Then*

$$\mathcal{E}_a(f,f^2,\ldots,f^k)=\mathcal{E}_a(g,g^2,\ldots,g^k)$$

if and only if there exists some permutation $\sigma:[1,k]\longrightarrow[1,k]$ such that $g^i(a)=f^{\sigma(j)}(a)$ for any $1\leq i\leq j\leq k$.

Proposition 3.2. *Let f induce a dynamical system on $a \in \mathbb{N}$. If the s^{th} dynamical system is stable relative to the $(s-1)^{th}$ dynamical system, then*

$$f^{s}(a) = f^{s+1}(a) = \ldots = f^{s+l}(a) = \ldots$$

for all $l \geq 1$.

Proof. Suppose the s^{th} dynamical system is stable relative to the $(s-1)^{th}$ dynamical system, then $f^{s}(a) = f^{s-1}(a)$ so that we have the chain of equality

$$f^{s}(a) = f^{s+1}(a) = f^{s+2}(a) = \ldots = f^{s+3}(a) = \ldots$$

by iteration. □

3.1. **Analysis on dynamical balls.** In this section we develop some topology of dynamical balls and study their interaction with each other. We launch the following languages in the sequel.

Definition 3.2. Let $\mathcal{B}_{f^{j}(a)}(a)$ be the j-dynamical ball induced by the k-dynamical system $f(a), f^{2}(a), f^{3}(a), \ldots, f^{k}(a)$ with $1 \leq j \leq k$. Then we say $y_n \in \mathcal{B}_{f^{j}(a)}(a)$ if and only if the inequality holds

$$|y_n - a| < f^{j}(a).$$

In particular we write $y_n \hat{\in} \mathcal{B}_{f^{j}(a)}(a)$ if and only if $|y_n - a| = f^{j}(a)$.

Proposition 3.3. *Let $x_n \in \mathcal{B}_{f^{j}(a)}(a)$ and $y_n \in \mathcal{B}_{f^{i}(a)}(a)$ with $1 \leq i < j \leq k$. Then the following holds*

- (i) $x_n - y_n \in \mathcal{B}_{f^{i}(af^{s}(a))}(a)$ *provided* $f^{i}(a) + f^{i}(f^{s}(a)) \leq f^{i}(af^{s}(a))$ *for* $s = j - i$
- (ii) $x_n - y_n \in \mathcal{B}_{f^{i}(a+f^{s}(a))}(a)$ *provided* $f^{i}(a) + f^{i}(f^{s}(a)) \leq f^{i}(a + f^{s}(a))$ *for* $s = j - i$.

Proof. The following containment $x_n \in \mathcal{B}_{f^{j}(a)}(a)$ and $y_n \in \mathcal{B}_{f^{i}(a)}(a)$ implies that $|x_n - a| \leq f^{j}(a)$ and $|y_n - a| \leq f^{i}(a)$ so that with $1 \leq i < j \leq k$, we can write the inequality

$$|x_n - y_n - a| \leq f^{j}(a) + f^{i}(a) = f^{i}(f^{s}(a)) + f^{i}(a)$$

and (i) and (ii) follows under the specified requirements. □

Proposition 3.4. *Let*

$$f(a), f^{2}(a), f^{3}(a), \ldots, f^{k}(a)$$

be a k-dynamical system with corresponding sequence of dynamical balls

$$\mathcal{B}_{f(a)}(a), \mathcal{B}_{f^{2}(a)}(a), \ldots, \mathcal{B}_{f^{k}(a)}(a).$$

Then the following embedding holds

$$\mathcal{B}_{f^{j}(a)}(a) \subseteq \mathcal{B}_{f^{j+1}(a)}(a)$$

if and only $f^{j}(a) \leq f^{j+1}(a)$

This is an easy consequence of the interpretation of the sequence of dynamical balls all centered at a fixed $a \in \mathbb{N}$ and evolves according to the radius whose values are terms in the corresponding induced dynamical system induced on $a \in \mathbb{N}$ by $f : \mathbb{N} \longrightarrow \mathbb{N}$.

3.2. **The limit of dynamical balls.** In this subsection we introduce and study the notion of the limit of dynamical balls.

Definition 3.3. Let

$$f(a), f^2(a), f^3(a), \ldots, f^k(a)$$

be a k-dynamical system with corresponding sequence of dynamical balls

$$\mathcal{B}_{f(a)}(a), \mathcal{B}_{f^2(a)}(a), \ldots, \mathcal{B}_{f^k(a)}(a).$$

Then we denote the limit of the dynamical balls as $\lim_{k\longrightarrow\infty} \mathcal{B}_{f^k(a)}(a)$. We write

$$\lim_{k\longrightarrow\infty} \mathcal{B}_{f^k(a)}(a) = \mathcal{B}_b(a)$$

if and only if for any $\delta > 0$ and for any $x_n \hat{\in} \mathcal{B}_b(a)$ there exists some $K_o > 0$ such that for all $k \geq K_o$ there exists some $y_n \hat{\in} \mathcal{B}_{f^k(a)}(a)$ such that

$$|x_n - y_n| < \delta$$

and we say the sequence of k-dynamical balls **converges**. Otherwise we say it **diverges**.

Let

$$f(a), f^2(a), f^3(a), \ldots, f^k(a)$$

be a k-dynamical system with corresponding sequence of dynamical balls

$$\mathcal{B}_{f(a)}(a), \mathcal{B}_{f^2(a)}(a), \ldots, \mathcal{B}_{f^k(a)}(a).$$

Then the following statements are equivalent:
For any $\epsilon > 0$ there exists some $x_n \hat{\in} \mathcal{B}_{f^s(a)}(a)$ and $y_n \hat{\in} \mathcal{B}_{f^{s-1}(a)}(a)$ such that

$$|x_n - y_n| < \epsilon$$

for $1 \leq s \leq k$ as $k \longrightarrow \infty$ if and only if the corresponding infinite dynamical system

$$f(a), f^2(a), f^3(a), \ldots, f^k(a), f^{k+1}(a), \ldots$$

is a Cauchy sequence. Let us suppose the corresponding infinite dynamical system is a Cauchy sequence, then it follows that for any $\epsilon > 0$ there exists a $K_o > 0$ such that for all $s \geq K_o$ then

$$|f^s(a) - f^{s-1}(a)| = |a + f^s(a) - (a + f^{s-1}(a))| < \epsilon$$

so that by taking $x_n = a + f^s(a)$ and $y_n = a + f^{s-1}(a)$ we see that $x_n \hat{\in} \mathcal{B}_{f^s(a)}(a)$ and $y_n \hat{\in} \mathcal{B}_{f^{s-1}(a)}(a)$. Conversely suppose that for any $\epsilon > 0$ there exists some $x_n \hat{\in} \mathcal{B}_{f^s(a)}(a)$ and $y_n \hat{\in} \mathcal{B}_{f^{s-1}(a)}(a)$ such that

$$|x_n - y_n| < \epsilon$$

for $1 \leq s \leq k$ as $k \longrightarrow \infty$. By taking $x_n = a + f^s(a) \hat{\in} \mathcal{B}_{f^s(a)}(a)$ and $y_n = a + f^{s-1}(a) \hat{\in} \mathcal{B}_{f^{s-1}(a)}(a)$, we see that

$$|f^s(a) - f^{s-1}(a)| < \epsilon$$

for $1 \leq s \leq k$ as $k \longrightarrow \infty$.

Proposition 3.5. *Let $\mathcal{B}_{f^j(a)}(a)$ be a dynamical ball induced by the k-dynamical system $f(a), f^2(a), f^3(a), \ldots, f^k(a)$ with $1 \leq j \leq k$. Then for any $\epsilon > 0$ there exists some $x_n \hat{\in} \mathcal{B}_{f^s(a)}(a)$ and $y_n \hat{\in} \mathcal{B}_{f^{s-1}(a)}(a)$ such that*

$$|x_n - y_n| < \epsilon$$

for $1 \leq s \leq k$ as $k \longrightarrow \infty$ if and only if $\lim\limits_{j \longrightarrow \infty} \mathcal{B}_{f^j(a)}(a)$ exists.

Proof. Let $\epsilon > 0$ and suppose there exists some $x_n \hat{\in} \mathcal{B}_{f^s(a)}(a)$ and $y_n \hat{\in} \mathcal{B}_{f^{s-1}(a)}(a)$ such that

$$|x_n - y_n| < \epsilon$$

for $1 \leq s \leq k$ as $k \longrightarrow \infty$. Then it implies that the infinite dynamical system

$$f(a), f^2(a), \ldots, f^k(a), \ldots,$$

must be a Cauchy sequence, so that there exists some $L \in \mathbb{R}^+$ such that

$$\lim_{j \longrightarrow \infty} f^j(a) = L.$$

It follows that $\lim\limits_{j \longrightarrow \infty} \mathcal{B}_{f^j(a)}(a)$ exists and

$$\lim_{j \longrightarrow \infty} \mathcal{B}_{f^j(a)}(a) = \mathcal{B}_L(a).$$

Conversely, suppose $\lim\limits_{j \longrightarrow \infty} \mathcal{B}_{f^j(a)}(a)$ exists and let

$$\lim_{j \longrightarrow \infty} \mathcal{B}_{f^j(a)}(a) = \mathcal{B}_L(a).$$

Then it follows that for any $\epsilon > 0$ and for any $b \hat{\in} \mathcal{B}_L(a)$ there exists some $K_o > 0$ such that for all $s \geq K_o$ then there exists some $x_n \hat{\in} \mathcal{B}_{f^s(a)}(a)$ such that $|x_n - b| < \frac{\epsilon}{2}$. It follows similarly that there exists some $y_n \hat{\in} \mathcal{B}_{f^{s-1}(a)}(a)$ such that $|y_n - b| < \frac{\epsilon}{2}$ so that for all $s \geq K_o$

$$|x_n - y_n| \leq |x_n - b| + |y_n - b| < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon.$$

□

3.3. **Dynamical waves and amplitude of waves induced by dynamical balls.** In this section we introduce and study the notion of dynamical waves and their corresponding notion of amplitudes induced by the evolution of dynamical balls.

Definition 3.4. Let $\mathcal{B}_{f^j(a)}(a)$ be a dynamical ball induced by the k-dynamical system $f(a), f^2(a), f^3(a), \ldots, f^k(a)$ with $1 \leq j \leq k$. Then we call the sequence of discrepancy

$$(|f^{j+1}(a) - f^j(a)|)_{1 \leq j \leq k}$$

the dynamical **waves** induced by the evolution of the dynamical balls. We call each term of the sequence a **wavelet** of the dynamical system. We call $\sup_{1 \leq j \leq k}(|f^{j+1}(a) - f^j(a)|)$ the **amplitude** of the wave and we denote amplitude by $\mathbb{A}_f(a,k)$.

Definition 3.5. Let

$$f(a), f^2(a), f^3(a), \ldots, f^k(a)$$

be a k-dynamical system with corresponding sequence of dynamical balls

$$\mathcal{B}_{f(a)}(a), \mathcal{B}_{f^2(a)}(a), \ldots, \mathcal{B}_{f^k(a)}(a).$$

By the **frequency** of the dynamical **wave** induced, we mean the formal sum

$$\mathcal{W}_a(f,k) := \sum_{j=1}^{k} \frac{|f^{j+1}(a) - f^j(a)|}{j}.$$

We denote the **frequency** of the wave of the corresponding infinite dynamical system as

$$\mathcal{W}_a(f) = \lim_{k \longrightarrow \infty} \sum_{j=1}^{k} \frac{|f^{j+1}(a) - f^j(a)|}{j} = \sum_{j=1}^{\infty} \frac{|f^{j+1}(a) - f^j(a)|}{j}.$$

It turns out that for any dynamical wave induced by f on $a \in \mathbb{N}$, we can decompose the total dynamical waves into two pieces, namely as small piece and a large piece as follows

$$\begin{aligned}\mathcal{D}_f(a,k) : &= \sum_{2 \leq s \leq k} |f^s(a) - f^{s-1}(a)| \\ &= \sum_{\substack{2 \leq s \leq k \\ |f^s(a) - f^{s-1}(a)| > |f^2(a) - f(a)|}} |f^s(a) - f^{s-1}(a)| \\ &+ \sum_{\substack{2 \leq s \leq k \\ |f^s(a) - f^{s-1}(a)| < |f^2(a) - f(a)|}} |f^s(a) - f^{s-1}(a)|\end{aligned}$$

and we call the second sum on the right-hand side the **regular** part and the first sum the **random** part. Symbolically, we rewrite the above decomposition into random and regular part as

$$\mathcal{D}_f(a,k) := \mathbb{R}ad_f(a,k) + \mathbb{R}eg_f(a,k).$$

It is easy to see that for any dynamical system, we can write

$$\mathcal{D}_f(a,k) := \sum_{\substack{2 \leq s \leq k \\ |f^s(a) - f^{s-1}(a)| > |f^2(a) - f(a)|}} |f^s(a) - f^{s-1}(a)| + O_{f,a}(k).$$

The corresponding total wave of the infinite dynamical system is obtained by taking the limits

$$\begin{aligned}\mathcal{D}_f(a) &= \lim_{k \longrightarrow \infty} \mathcal{D}_f(a,k) \\ &= \sum_{s=2}^{\infty} |f^s(a) - f^{s-1}(a)|.\end{aligned}$$

Proposition 3.6. *Let*

$$f(a), f^2(a), f^3(a), \ldots, f^k(a)$$

be a k-dynamical system with corresponding sequence of dynamical balls

$$\mathcal{B}_{f(a)}(a), \mathcal{B}_{f^2(a)}(a), \ldots, \mathcal{B}_{f^k(a)}(a).$$

Then $\mathcal{W}_a(f) < \infty$ *if and only if* $\lim\limits_{j \longrightarrow \infty} \mathcal{B}_{f^j(a)}(a)$ *exists.*

Proof. Suppose $\mathcal{W}_a(f) < \infty$, then it follows that

$$\sum_{j=1}^{\infty} \frac{|f^{j+1}(a) - f^j(a)|}{j} < \infty$$

so that $\lim\limits_{j \longrightarrow \infty} |f^{j+1}(a) - f^j(a)| = 0$ and it implies that $\lim\limits_{j \longrightarrow \infty} f^j(a)$ exists and so is $\lim\limits_{j \longrightarrow \infty} \mathcal{B}_{f^j(a)}(a)$. Conversely suppose $\lim\limits_{j \longrightarrow \infty} \mathcal{B}_{f^j(a)}(a)$ exists then so is $\lim\limits_{j \longrightarrow \infty} f^j(a)$ so that $\lim\limits_{j \longrightarrow \infty} |f^{j+1}(a) - f^j(a)| = 0$ and $\mathcal{W}_a(f) < \infty$. ☐

Theorem 3.6 (Restriction law)**.** *Let*

$$f(a), f^2(a), f^3(a), \ldots, f^k(a)$$

be a k-dynamical system such that $|f^{s+1}(a) - f^s(a)| \neq |f^{t+1}(a) - f^t(a)|$ *for all* $s, t \geq 1$ *with* $s \neq t$ *and* $|f^{s+1}(a) - f^s(a)|, |f^{t+1}(a) - f^t(a)| \neq 0$ *with corresponding sequence of dynamical balls*

$$\mathcal{B}_{f(a)}(a), \mathcal{B}_{f^2(a)}(a), \ldots, \mathcal{B}_{f^k(a)}(a).$$

Then $\lim\limits_{k \longrightarrow \infty} \mathbb{R}eg_f(a, k) < \infty$.

Proof. Let us suppose on the contrary that $\lim\limits_{k \longrightarrow \infty} \mathbb{R}eg_f(a, k) = \infty$ so that

$$\lim_{k \longrightarrow \infty} \sum_{\substack{2 \leq s \leq k \\ |f^s(a) - f^{s-1}(a)| < |f^2(a) - f(a)|}} |f^s(a) - f^{s-1}(a)|$$

contains infinitely many terms. It follows from the condition $|f^s(a) - f^{s-1}(a)| < |f^2(a) - f(a)|$ and the pigeon hole principle that there are infinitely many coinciding wavelets. It follows that there must exist some $s \neq t$ such that $|f^{s+1}(a) - f^s(a)| = |f^{t+1}(a) - f^t(a)|$. This contradicts the requirements of the dynamical system induced. ☐

Remark 3.7. Theorem 3.6 though simple is ultimately useful for determining the convergence of dynamical systems. The bound on the regular part of the total wave of any infinite dynamical system reduces the problem of convergence to just the random part of the decomposition. These ideas are summarized in the following proposition.

Proposition 3.7. *Let*

$$f(a), f^2(a), f^3(a), \ldots, f^k(a)$$

be a k-dynamical system such that $|f^{s+1}(a) - f^s(a)| \neq |f^{t+1}(a) - f^t(a)|$ *for all* $s, t \geq 1$ *with* $s \neq t$ *and* $|f^{s+1}(a) - f^s(a)|, |f^{t+1}(a) - f^t(a)| \neq 0$ *with corresponding sequence of dynamical balls*

$$\mathcal{B}_{f(a)}(a), \mathcal{B}_{f^2(a)}(a), \ldots, \mathcal{B}_{f^k(a)}(a).$$

Then $\lim\limits_{j \longrightarrow \infty} \mathcal{B}_{f^j(a)}(a)$ *exists if and only if* $\lim\limits_{k \longrightarrow \infty} \mathbb{R}ad_f(a, k) < \infty$.

Proof. Suppose that $\lim\limits_{j\longrightarrow\infty}\mathcal{B}_{f^j(a)}(a)$ exists then it follows that $\lim\limits_{j\longrightarrow\infty}|f^j(a)-f^{j-1}(a)|=0$. It implies that

$$\lim_{k\longrightarrow\infty}\mathbb{R}ad_f(a,k)<\mathcal{D}_f(a)<\infty.$$

Conversely suppose $\lim\limits_{k\longrightarrow\infty}\mathbb{R}ad_f(a,k)<\infty$, then by appealing to Theorem 3.6 we can write

$$\lim_{k\longrightarrow\infty}\mathcal{D}_f(a,k)=\lim_{k\longrightarrow\infty}\mathbb{R}ad_f(a,k)+\lim_{k\longrightarrow\infty}\mathbb{R}eg_f(a,k)<\infty$$

so that

$$\sum_{s=1}^{\infty}|f^s(a)-f^{s-1}(a)|<\infty.$$

Since $f^j(a)-f^{j-1}(a)\in\mathbb{Z}$, it implies that $\lim\limits_{j\longrightarrow\infty}|f^j(a)-f^{j-1}(a)|=0$ and that $\lim\limits_{j\longrightarrow\infty}\mathcal{B}_{f^j(a)}(a)$ also exists. □

3.4. **Dynamical waves estimate.** In this section we establish some new estimates relating the frequency, amplitude and total waves of any dynamical systems.

Theorem 3.8. *Let*

$$f(a),f^2(a),f^3(a),\ldots,f^k(a)$$

be a k-dynamical system such that $|f^{s+1}(a)-f^s(a)|\neq|f^{t+1}(a)-f^t(a)|$ for all $s,t\geq 1$ with $s\neq t$ and $|f^{s+1}(a)-f^s(a)|,|f^{t+1}(a)-f^t(a)|\neq 0$ with corresponding sequence of dynamical balls

$$\mathcal{B}_{f(a)}(a),\mathcal{B}_{f^2(a)}(a),\ldots,\mathcal{B}_{f^k(a)}(a).$$

Then

(i)
$$\mathcal{W}_f(a,k)\ll\mathbb{A}_f(a,k)\log k$$

(ii)
$$\int\limits_{1}^{k-1}\frac{f^t(a)}{t^2}dt\ll_{f,a}\mathcal{W}_f(a,k)-\frac{\mathcal{D}_f(a,k)}{k}$$

(iii)
$$\mathcal{W}_f(a,k)=\frac{\mathbb{R}ad_f(a,k)}{k}+\int\limits_{1}^{k-1}\frac{\mathbb{R}ad_f(a,t)}{t^2}dt+O(\frac{1}{k})$$

(iv)
$$\int\limits_{1}^{k-1}\frac{|f^t(a)-f(a)|}{t^2}dt\leq\int\limits_{1}^{k-1}\frac{\mathbb{R}ad_f(a,t)}{t^2}dt+O(\frac{1}{k})$$

Proof. (i) We can write

$$\mathcal{W}_f(a,k) = \sum_{j=1}^{k-1} \frac{|f^{j+1}(a) - f^j(a)|}{j}$$
$$\leq \mathbb{A}_f(a,k) \sum_{j=1}^{k-1} \frac{1}{j} \ll \mathbb{A}_f(a,k) \log k.$$

(ii) By an application of partial summation, we can write the frequency of the dynamical wave

$$\mathcal{W}_f(a,k) = \frac{1}{k}\mathcal{D}_f(a,k) + \int\limits_1^{k-1} \frac{\mathcal{D}_f(a,t)}{t^2} dt$$

so that by exploiting the lower bound

$$\mathcal{D}_f(a,t) := \sum_{1 \leq j \leq t} |f^{j+1}(a) - f^j(a)| \geq |f^t(a) - f(a)| \gg_{f,a} f^t(a)$$

the asserted estimate follows.

(iii) By applying the decomposition of the total dynamical waves into random and the regular part as $\mathcal{D}_f(a,k) = \mathbb{R}ad_f(a,k) + \mathbb{R}eg_f(a,k)$, we can further write the estimate for the frequency in (ii) as

$$\begin{aligned}\mathcal{W}_f(a,k) &= \frac{1}{k}\mathcal{D}_f(a,k) + \int\limits_1^{k-1} \frac{\mathcal{D}_f(a,t)}{t^2} dt \\ &= \frac{1}{k}\mathbb{R}ad_f(a,k) + \frac{1}{k}\mathbb{R}eg_f(a,k) + \int\limits_1^{k-1} \frac{\mathbb{R}ad_f(a,t)}{t^2} dt + \int\limits_1^{k-1} \frac{\mathbb{R}eg_f(a,t)}{t^2} dt \\ &= \frac{\mathbb{R}ad_f(a,k)}{k} + \int\limits_1^{k-1} \frac{\mathbb{R}ad_f(a,t)}{t^2} dt + O(\frac{1}{k})\end{aligned}$$

since

$$\mathbb{R}eg_f(a,k) \ll_{f,a} 1$$

and

$$\int\limits_1^{k-1} \frac{\mathbb{R}eg_f(a,t)}{t^2} dt \leq \int\limits_1^{\infty} \frac{\mathbb{R}eg_f(a,t)}{t^2} dt \ll_{f,a} \int\limits_1^{\infty} \frac{1}{t^2} dt$$

.

(iv) By plugging the estimate in (iii) into the upper bound

$$\int\limits_1^{k-1} \frac{|f^t(a) - f(a)|}{t^2} dt \leq \mathcal{W}_f(a,k) - \frac{\mathcal{D}_f(a,k)}{k}$$

the claimed upper bound also follows.

□

The estimates established in Theorem 3.8 can be used in a unifying manner to study the convergence of any dynamical system. The estimate in (iii) seems to stand out among them and confirms Proposition 3.7. We confirm the observation again as an application of the estimate.

Corollary 3.1. Let

$$f(a), f^2(a), f^3(a), \ldots, f^k(a)$$

be a k-dynamical system such that $|f^{s+1}(a) - f^s(a)| \neq |f^{t+1}(a) - f^t(a)|$ for all $s, t \geq 1$ with $s \neq t$ and $|f^{s+1}(a) - f^s(a)|, |f^{t+1}(a) - f^t(a)| \neq 0$ with corresponding sequence of dynamical balls

$$\mathcal{B}_{f(a)}(a), \mathcal{B}_{f^2(a)}(a), \ldots, \mathcal{B}_{f^k(a)}(a).$$

Then $\lim\limits_{j \longrightarrow \infty} \mathcal{B}_{f^j(a)}(a)$ exists if and only if $\lim\limits_{k \longrightarrow \infty} \mathbb{R}ad_f(a,k) < \infty$.

Proof. The result follows from the estimate

$$\mathcal{W}_f(a,k) = \frac{\mathbb{R}ad_f(a,k)}{k} + \int\limits_{1}^{k-1} \frac{\mathbb{R}ad_f(a,t)}{t^2} dt + O(\frac{1}{k})$$

and an appeal to Proposition 3.6. □

3.5. **Translation and dilation of dynamical balls.** In this section we introduce the notion of translation and dilation of dynamical balls. This would allow the movement of dynamical balls for the purposes of our work.

Definition 3.9. Let

$$f(a), f^2(a), f^3(a), \ldots, f^k(a)$$

be a k-dynamical system with corresponding sequence of dynamical balls

$$\mathcal{B}_{f(a)}(a), \mathcal{B}_{f^2(a)}(a), \ldots, \mathcal{B}_{f^k(a)}(a).$$

We call the map

$$\mathbb{T}_b : \mathcal{B}_{f^j(a)}(a) \longmapsto \mathcal{B}_{f^j(a+b)}(a+b) := \mathcal{B}_{f^j(\mathbb{T}_b(a))}(\mathbb{T}_b(a))$$

the **translation** of the dynamical ball $\mathcal{B}_{f^j(a)}(a)$ by a a scale factor b.

Definition 3.10. Let

$$f(a), f^2(a), f^3(a), \ldots, f^k(a)$$

be a k-dynamical system with corresponding sequence of dynamical balls

$$\mathcal{B}_{f(a)}(a), \mathcal{B}_{f^2(a)}(a), \ldots, \mathcal{B}_{f^k(a)}(a).$$

We call the map

$$\mathbb{D}_m : \mathcal{B}_{f^j(a)}(a) \longmapsto \mathcal{B}_{f^j(ma)}(ma) := \mathcal{B}_{f^j(\mathbb{D}_m(a))}(\mathbb{D}_m(a))$$

the **dilation** of the dynamical ball $\mathcal{B}_{f^j(a)}(a)$ by a scale factor m.

Proposition 3.8. *Let*

$$f(a), f^2(a), f^3(a), \ldots, f^k(a)$$

be a k-dynamical system with corresponding sequence of dynamical balls

$$\mathcal{B}_{f(a)}(a), \mathcal{B}_{f^2(a)}(a), \ldots, \mathcal{B}_{f^k(a)}(a).$$

Suppose $\lim\limits_{j\longrightarrow\infty}\mathcal{B}_{f^j(b)}(b)$ *exists. If* $\lim\limits_{j\longrightarrow\infty}\mathcal{B}_{f^j(a)}(a)$ *exists then* $\lim\limits_{j\longrightarrow\infty}\mathcal{B}_{f^j(a+b)}(a+b)$ *exists provided* $f^s(a+b) \leq f^s(a)+f^s(b)$ *whenever* $f^{s-1}(a+b) \geq f^{s-1}(a)+f^{s-1}(b)$ *for all* $s \geq 2$.

Proof. It suffices to show that for any $\epsilon > 0$ there exists some $N_o > 0$ such that for all $s \geq N_o$ then $|f^s(a+b) - f^{s-1}(a+b)| < \epsilon$.
Under the assumption $\lim\limits_{j\longrightarrow\infty}\mathcal{B}_{f^j(b)}(b)$ and $\lim\limits_{j\longrightarrow\infty}\mathcal{B}_{f^j(a)}(a)$ exist, then for any $\epsilon > 0$ there exist some $N_o, M_o > 0$ such that

$$|f^s(a) - f^{s-1}(a)| < \frac{\epsilon}{2}$$

for all $s \geq N_o$ and

$$|f^s(b) - f^{s-1}(b)| < \frac{\epsilon}{2}$$

for all $s \geq M_o$. By choosing $P = \max\{N_o, M_o\}$ and exploiting the condition $f^s(a+b) \leq f^s(a)+f^s(b)$ if $f^{s-1}(a+b) \geq f^{s-1}(a)+f^{s-1}(b)$ for all $s \geq 2$, it follows that

$$|f^s(a+b) - f^{s-1}(a+b)| \leq |f^s(a) - f^{s-1}(a)| + |f^s(b) - f^{s-1}(b)| < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon$$

for all $s \geq P = \max\{N_o, M_o\}$. This implies that $\lim\limits_{j\longrightarrow\infty}\mathcal{B}_{f^j(a+b)}(a+b)$ exists since $\epsilon > 0$ can be chosen arbitrarily. □

Proposition 3.9. *Let*

$$f(a), f^2(a), f^3(a), \ldots, f^k(a)$$

be a k-dynamical system with corresponding sequence of dynamical balls

$$\mathcal{B}_{f(a)}(a), \mathcal{B}_{f^2(a)}(a), \ldots, \mathcal{B}_{f^k(a)}(a).$$

If $\lim\limits_{j\longrightarrow\infty}\mathcal{B}_{f^j(a)}(a)$ *exists then* $\lim\limits_{j\longrightarrow\infty}\mathcal{B}_{f^j(ma)}(ma)$ *exists provided* $f^s(ma) \leq mf^s(a)$ *whenever* $f^{s-1}(ma) \geq mf^{s-1}(a)$ *for all* $s \geq 2$ *and for a fixed* $m \in \mathbb{N}$.

Proof. Under the assumption $\lim\limits_{j\longrightarrow\infty}\mathcal{B}_{f^j(a)}(a)$ exists, then for any $\epsilon > 0$ there exists some $N_o > 0$ such that

$$|f^s(a) - f^{s-1}(a)| < \epsilon$$

for all $s \geq N_o$ so that under the conditions $f^s(ma) \leq mf^s(a)$ whenever $f^{s-1}(ma) \geq mf^{s-1}(a)$ for all $s \geq 2$ and for a fixed $m \in \mathbb{N}$, we can write by choosing $\epsilon = \frac{\delta}{m}$ for any $\delta > 0$

$$|f^s(ma) - f^{s-1}(ma)| \leq m|f^s(a) - f^{s-1}(a)| < m\epsilon = \delta$$

for all $s \geq N_o$. □

Remark 3.11. Proposition 3.8 and 3.9 provides a slick way of extending the convergence of an infinite dynamical system induced by a function f on any $a \in \mathbb{N}$ to some other numbers $z \in \mathbb{N}$ by translation.

[1].

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